ELEMENTS

OF

GEOMETRY

WITH EXERCISES FOR STUDENTS, AND AN INTRODUCTION TO MODERN GEOMETRY.

BY

A. SCHUYLER, LL.D.,

President of Baldwin University, Author of Higher Arithmetic, Principles of Logic, Complete Algebra, Surveying and Navigation, and Trigonometry and Mensuration.

WILSON, HINKLE & CO.,

137 WALNUT STREET
CINCINNATI

28 BOND STREET
NEW YORK

GRADED-SCHOOL SERIES.

HARVEY'S SPELLER AND READERS.
HARVEY'S LANGUAGE COURSE.
WHITE'S ARITHMETICS.
ECLECTIC GEOGRAPHIES.
SCHUYLER'S ALGEBRA AND GEOMETRY.

ELECTROTYPED AT
FRANKLIN TYPE FOUNDRY
CINCINNATI

ECLECTIC PRESS
WILSON, HINKLE & CO
CINCINNATI

PREFACE.

A NEW treatise on Geometry, to be of sufficient merit to claim attention, must be both conservative and progressive. It should lay firm hold on the past, embody the present state of the science, and anticipate future developments. A work claiming to be wholly new might, perhaps with justice, be at once discarded as worthless; while one containing no improvements could not justify its own existence.

The geometrical objects, — points, lines, surfaces, solids, and angles, — constitute the subject-matter of the science; the definitions are the tests by which these objects are discriminated and their classification determined; the axioms are the warrants for the steps taken in the course of demonstration; the postulates justify the assumption of magnitudes having any position, form, and extent.

The logical principles which underlie the demonstrations of this volume have been carefully discriminated and illustrated. The discussion of the axioms and postulates is the result of research, and intent and prolonged thought. That fundamental principles have been reached is manifest from their underivability, and the simplicity of the deduction of the ordinary so-called axioms from them as corollaries.

Mr. Bain has observed of the principle, If A be greater than B, and B greater than C, much more is A greater than C: "If it can not be deductively inferred from the proper axioms, it will have to be received as a third axiom." Not only can this principle be inferred (**23,** 20), but also Mr. Bain's so-called proper axioms (**23,** 3, 6).

The proposition, "A straight line is the shortest distance from one point to another," has, by some mathematicians, been given as a definition; by others, as an axiom. This discrepancy raises the question whether it is either. A straight line is not defined by calling it the shortest distance from one point to another; for this does not, as a correct definition would, develop the idea of a straight line; but assuming the idea developed, it affirms of the straight line an additional truth, that any portion of it is the shortest distance from one of the points limiting that portion to the other point. But this truth is demonstrable (**67**), and is, therefore, not an axiom, but a theorem.

The discussion of parallels, a matter involved in controversy, requires no additional axiom — an evidence of the sufficiency of the axioms as given, and a justification of the definition of parallels.

The theory of limits has been applied to the measurement of the circle, thus securing demonstrations irreproachable in their logical character.

When a reference is given, the student should state the principle involved; when the query mark (?), which is to be interpreted "why?" is used, he should give the reason.

The discussion of proportion, as properly belonging to Algebra, has been omitted. The references to that subject are to the author's COMPLETE ALGEBRA.

Book VIII treats of Modern Geometry, which has within the last century so greatly stimulated the growth of Elementary Geometry, for a time arrested by the analytic method of Descartes.

By omitting the Supplementary Sections and the Modern Geometry, Book VIII, those schools that wish a moderate course can be accommodated in a simple manner.

The Exercises, always interesting to the student, can be taken, one or two each day, while the class is pursuing the next section.

Thanks are due to Prof. Warner for invaluable aid.

Hoping that the work will receive the candid attention of mathematicians, and the favorable consideration of the public, it is sent forth to accomplish its mission.

CONTENTS.

ELEMENTS OF GEOMETRY

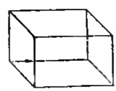
INTRODUCTION.

I. GENERAL OUTLINE.

1. Geometrical Objects.

1. **Space** is indefinite extension in all directions.

2. A **geometric solid** is a portion of space having three dimensions, — length, breadth, thickness — and a definite form. In Geometry, the term *solid* is used for *geometric solid*. The annexed diagram, which is a representation of a solid, will make the following statements clear:



3. The limits or boundaries of a solid are *surfaces*.

4. The limits or boundaries of a snrface are *lines*.

5. The limits or extremities of a line are *points*.

6. The intersection of two surfaces is a *line*.

7. The intersection of two lines is a *point*.

8. The divergence or mutual inclination of lines or surfaces is an *angle*.

9. A line can be generated by the movement of a point; that is, a line is the path traced by a moving point.

10. A surface can be generated by the movement of a line.

11. A solid can be generated by the movement of a surface.

12. An angle can be generated by the revolntion of one of two lines or surfaces, from a position of coincidence with the other, about a common point or line.

Points, lines, surfaces, and solids, may be defined thus:

13. A **point** is position without extension.

14. A **line** is extension in one dimension,—*length.*

15. A **surface** is extension in two dimensions,—*length* and *breadth.*

16. A **solid** is extension in three dimensions,—*length, breadth,* and *thickness.*

17. Points have *position* only; but lines, surfaces, solids, and angles have *position, magnitude,* and *form.*

18. **Geometry** is the science of position, magnitude, and form.

2. Points.

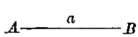
1. A **point** may be regarded, 1st, as a limit of a line; 2d, as the intersection of two lines; or, 3d, without reference to lines, as position in space, without magnitude.

2. A point is designated by a letter, and read by reading the letter.

3. Lines.

1. A **line** may be regarded, 1st, as a limit of a surface; 2d, as the intersection of two surfaces; 3d, as the path generated by the movement of a point; 4th, as the assemblage of all the positions of the generating point; or, 5th, without reference to surfaces or points, as extension in one dimension,—length, without breadth or thickness.

2. A line may be designated by letters placed near two of its points, or by a small italic letter near its side, and be read by reading the letters or letter. Thus, the line *AB*, or the line *a*.



3. Lines may be classified as *elementary* and *composite.*

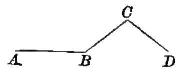
4. **Elementary lines** are classified as *straight* and *curved.*

5. A **straight line** is the trace of a point moving continuously in the same direction.

6. A **curved line** is the trace of a moving point which continually changes its direction.

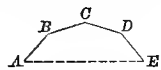
7. **Composite lines** are classified as *broken* and *mixed*.

8. A **broken line** is a series of connected straight lines, called its components, lying in different directions.



Thus, $ABCD$ is a broken line, and AB, BC, CD, are its components.

9. A **convex broken line** is a broken line no component of which can, if produced, enter the space inclosed by the broken line and the straight line joining its extremities.



Thus, $ABCDE$ is a convex broken line.

10. A **mixed line** is a combination of straight and curved lines.

Thus, the last figure below is a mixed line.

11. Summary classification of lines:

- Lines
  - Elementary
    - Straight
    - Curved
  - Composite
    - Broken
    - Mixed

4. Surfaces.

1. A **surface** may be regarded 1st, as a limit of a solid; 2d, as the path generated by the movement of a line; or, 3d, without reference to solids or lines, as extension in two dimensions,—length and breadth, without thickness.

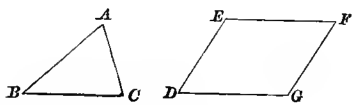
2. Surfaces may be classified as *elementary* and *composite*.

3. **Elementary surfaces** are classified as *plane* and *curved*.

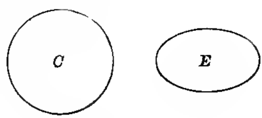
4. A **plane surface,** or simply a **plane,** is a surface such that the straight line joining any two of its points lies wholly in the surface.

5. Plane surfaces are classified with reference to their bounding lines, as *rectilinear*, *curvilinear*, and *mixtilinear*.

6. A **rectilinear surface,** called also a **plane polygon,** is a surface bounded by straight lines. The bounding lines are the *sides*, and the broken line forming the entire boundary is the *perimeter*.



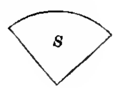
7. A polygon is represented by drawing its boundary, is designated by letters, and read by reading the letters. Thus, ABC and $DEFG$.



8. A **curvilinear surface** is a surface bounded by a curved line.

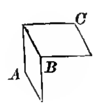
Thus, C, E, are curvilinear surfaces.

9. A **mixtilinear surface** is a surface bounded by a mixed line.



Thus, S is a mixtilinear surface.

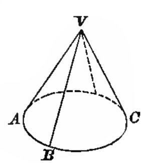
10. A surface is *curved* if any common intersection of it and a plane is a curved line.



11. **Composite surfaces** are classified as *broken* and *mixed*.

12. A **broken surface** is a surface composed of plane surfaces.

Thus, ABC is a broken surface.



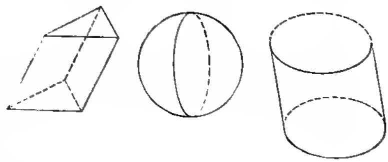
13. A **mixed surface** is a surface partly plane and partly curved.

Thus, $V\text{-}ABC$ is a mixed surface.

5. Solids.

1. A **solid** may be regarded, 1st, as the path generated by the movement of a plane; or, 2d, without reference to a plane, as extension in three dimensions,—length, breadth, and thickness.

2. Solids are classified thus: solids bounded by *plane surfaces*; solids bounded by *curved surfaces*; solids bounded by *mixed surfaces*.



6. Angles.

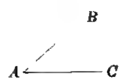
1. An **angle** is the divergence of intersecting lines or planes.

2. Angles are classified as *plane angles* and *solid angles.*

3. A **plane angle** is the divergence of two intersecting lines.

4. The diverging lines are called the *sides* of the angle.

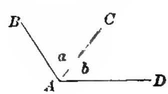
5. The point of intersection is called the *vertex* of the angle.



Thus, BAC is a plane angle; AB, AC, are its sides; A is its vertex.

6. An isolated angle may be read by reading the letter at its vertex; thus, A above. Angles having a common vertex are read by reading the three letters designating the sides of each, observing to read the letter at the vertex between the other two.

Thus, BAC, CAD, BAD.

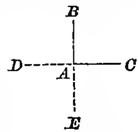


An angle may be designated by a small italic letter placed within, near the vertex; thus, a, b.

7. **Adjacent angles** are angles having a common vertex and a common side between them; thus, *BAC*, *CAD*, in the previous figure.

8. **Equal angles** are angles which can be made to coincide.

9. A **right angle** is an angle equal to the adjacent angle formed by producing either side through the vertex.



Thus, *BAC* is a right angle, if it is equal to *BAD* or to *CAE*, formed by producing *CA* or *BA*.

10. An **oblique angle** is an angle less or greater than a right angle.

11. Oblique angles are classified as *acute* and *obtuse.*

12. An **acute angle** is an oblique angle less than a right angle.

13. An **obtuse angle** is an oblique angle greater than a right angle.

7. Loci.

A **locus** is the place of all the points having a common property. Loci may be lines or surfaces.

8. Exercises.

1. Give a classification of surfaces, solids, angles. (3, 11)*

2. If the solid represented in the first article retain its length and breadth, and contract in thickness till it reaches its limit, what would it become?

3. If a surface retain its length and contract in breadth till it reaches its limit, what would it become?

4. If a line contract in length till it reaches its limit, what would it become?

---

* The reference is to Article 3, Paragraph 11. See page 11.

II. LOGICAL FACTS AND LAWS.

9. Definitions.

1. A **judgment** is the mental decision that a certain relation exists between two objects of thought.

2. A **proposition** is the expression of a judgment.

3. Propositions are classified in reference to form, as *categorical*, *hypothetical*, *disjunctive*, and *dilemmatic*.

4. A **categorical proposition** is a proposition not qualified by a condition. Thus, An acute angle is an oblique angle.

5. An **hypothetical proposition** is a proposition consisting of an hypothesis — a supposition or fact assumed — and a conclusion. Thus, If an angle is obtuse, it is oblique.

6. A **disjunctive proposition** is a proposition expressing an alternative. Thus, An oblique angle is acute or obtuse,

7. A **dilemmatic proposition** is a combination of an hypothetical and a disjunctive proposition. Thus, If an angle is oblique, it is acute or obtuse.

8. The *converse* of a *categorical* or a *disjunctive* proposition is the proposition obtained by interchanging the subject and predicate. Thus, No right angle is an oblique angle; conversely, No oblique angle is a right angle: also, An oblique angle is an acute or an obtuse angle; conversely, An acute or an obtuse angle is an oblique angle.

9. The *converse* of an *hypothetical* or a *dilemmatic* proposition is the proposition obtained by making the hypothesis of the original proposition the conclusion, and the conclusion the hypothesis. Thus, If an angle is right, it is not oblique; conversely, If an angle is oblique, it is not right: also, If an angle is oblique, it is acute or obtuse; conversely, If an angle is acute or obtuse, it is oblique.

10. Propositions are classified in reference to the relation of the subject and predicate, as *analytic* and *synthetic.*

11. An **analytic proposition** is a proposition in which the predicate expresses what is involved in the mere notion of the subject. Thus, A solid is extended.

12. A **synthetic proposition** is a proposition in which the predicate expresses something not involved in the mere notion of the subject. Thus, The sum of the two adjacent angles formed by one straight line meeting another is equal to two right angles.

13. Propositions are classified in reference to nature, as *definitions, axioms, absurdities, postulates, theorems, problems, lemmas, corollaries,* and *scholiums.*

14. A **definition** is such a description of an object as distinguishes it from all other objects. A definition of a class of objects includes every thing of that class, and excludes every thing else.

An object is defined by affirming it to belong to the class immediately containing it, and by distinguishing it from the rest of the class by stating its essential characteristics.

A definition is a categorical, analytic proposition.

The converse of a definition is true.

15. An **axiom** is a fundamental, self-evident truth.

An axiom is a synthetic proposition, fundamental and indemonstrable, and is necessarily, universally, and self-evidently true.

16. An **absurdity** is a self-evident falsity.

17. A **postulate** is a self-evident possibility. It asserts what evidently can be done, but does not tell how.

18. A **theorem** is a truth which requires proof.

19. A **formula** is a theorem expressed in algebraic language.

20. A **problem** is something proposed for solution.

21. A **corollary** is an obvious consequence.

22. A **scholium** is a note or remark.

23. A **lemma** is an auxiliary theorem or problem.

24. A **demonstration** is the proof of a proposition.

25. Demonstration is *direct* or *indirect*.

26. **Direct demonstration** proves a proposition in either of two ways:

1st. By superposition of one figure upon another.

2d. By logical combination of definitions, axioms, and propositions previously demonstrated.

27. **Indirect demonstration**, called also *reductio ad absurdum*, proves a proposition true by showing that the supposition that it is false involves a contradiction or an absurdity.

10. Relation of Propositions.

1. Any two propositions are *congruent* or *conflictive*.

2. **Congruent propositions** are those which are compatible. Thus, a and b are unequal, and a is less than b.

3. **Conflictive propositions** are those which are incompatible. Thus, a and b are equal, and a and b are unequal.

4. Conflictive propositions are *contraries* or *contradictories*.

5. **Contrary propositions** are conflictive propositions which are not universally inclusive. Thus, a is equal to b, and a is less than b, are contrary propositions; for they are conflictive, but are not universally inclusive, since another relation, a is greater than b, is possible.

6. **Contradictory propositions** are conflictive propositions which are universally inclusive. Thus, a and b are equal, and a and b are unequal, are contradictory propositions; for they are conflictive, since a and b can not be both equal and unequal; and they are universally inclusive, since a and b are either equal or unequal.

11. General Laws.

1. *Every proposition is either true or false.*

For, since truth and falsity are universally inclusive, no other case is possible.

2. *A proposition can not be both true and false.*

For, if so, it would be self-destructive.

3. *If a proposition is true, whatever it involves is also true.*

For the truth of a proposition necessitates the truth of what it involves.

12. Law of Congruents.

The congruity of two propositions is consistent with the truth of both, the falsity of both, or the truth of one and the falsity of the other.

Thus, a and b are unequal, and a is less than b, are congruents; and both true, if b is greater than a; both false, if a is equal to b; one true and the other false, if a is greater than b.

13. Corollary.

From mere congruence, the truth or falsity of one proposition does not involve the truth or falsity of another.

14. Law of Conflictives.

Two conflictive propositions can not both be true.

For, if so, they would mutually destroy each other, and both be false.

15. Corollaries.

1. *If one of two conflictives is true, the other is false.*

For, otherwise, we should have two conflictives both true, which is impossible.

2. *Two true propositions can not be conflictives.*

For, if so, we should have two conflictives both true, which is impossible.

3. *All truths exist in harmony.*

For, if two truths conflict with one another, the truth of either involves the falsity of the other, and each would be both true and false.

4. *A proposition is false if it involves the conflictive of a truth.*

For, if true, what it involves would be true, and we should have two conflictives both true, which is impossible.

16. Law of Contraries.

Contrary propositions can not both be true, but may both be false.

For, since they are conflictive, both can not be true; and since they are not universally inclusive, other cases are possible, and both may be false.

17. Corollaries.

1. *The truth of either of two contrary propositions involves the falsity of the other.*

For, since they are conflictive, both can not be true.

2. *The falsity of either of two contrary propositions does not involve the truth of the other.*

For, since they are not universally inclusive, other cases are possible, and both may be false.

18. Law of Contradictories.

One of two contradictory propositions must be true, and the other false.

For, since they are universally inclusive, no other case is possible, and one must be true; and, since they are conflictive, both can not be true.

19. Corollaries.

1. *The truth of either of two contradictory propositions involves the falsity of the other.*

For, since they are conflictives, both can not be true.

2. *The falsity of either of two contradictory propositions involves the truth of the other.*

For, since they are universally inclusive, no other case is possible, and one must be true.

20. Laws of Genera and Species.

1. *Inclusion in a species is inclusion in the genus.*

Thus, since both acute and obtuse angles are included as species in the genus oblique angles, if an angle is acute, it is oblique; or if an angle is obtuse, it is oblique.

2. *Inclusion in a species is exclusion from any thing from which the species is excluded.*

Thus, since acute angles are excluded from obtuse angles, if an angle is acute, it is not obtuse.

3. *Inclusion in the genus is not necessarily inclusion in a particular species.*

Thus, if an angle is oblique, it is not necessarily acute, for it may be obtuse; neither is it necessarily obtuse, for it may be acute.

4. *Inclusion in the genus is disjunctive inclusion in the species.*

Thus, if an angle is oblique, it is acute or obtuse.

5. *Exclusion from the genus is exclusion from all the species.*

Thus, if an angle is not oblique, it is neither acute nor obtuse.

6. *Exclusion from a species is not necessarily exclusion from the genus.*

Thus, if an angle is not acute, it is not necessarily not oblique; for it may be obtuse, and therefore oblique.

7. *Exclusion from all the species is exclusion from the genus.* Thus, if an angle is neither acute nor obtuse, it is not oblique.

8. *Whatever can be predicated, affirmatively or negatively, universally of any class, can in like manner be predicated of any thing contained in that class.*

III. AXIOMS AND POSTULATES.

21. Preliminaries.

1. **Similar magnitudes** are magnitudes identical in form.

2. **Equivalent magnitudes** are magnitudes identical in extent.

3. **Equal magnitudes** are magnitudes identical in form and extent.

4. Similar magnitudes may not be equivalent, and equivalent magnitudes may not be similar; but equal magnitudes are both similar and equivalent.

5. The **test** of the equality of two magnitudes is the fact that they can be made to coincide.

6. The **greater** of two unequal magnitudes is the one which exceeds the other in extent.

7. The **less** of two unequal magnitudes is the one which is exceeded by the other in extent.

8. The **whole** of a magnitude is all of it.

9. A **part** of a magnitude is any amount of it less than the whole.

10. The sum of all the parts taken in any order, and the sum of all the parts taken in any other order, are always equivalent, and, if similar, are equal.

11. A **substitute** for a magnitude is a magnitude which

will sustain to another magnitude the same relation as the given magnitude, if put in its place.

12. The **algebraic signs** will, in general, be employed in their usual sense; but the sign (=), which denotes either equality or equivalency, is, in a relation of equality, read, *is equal to*, or *equals*; but, in a relation of equivalency, *is equivalent to*.

22. Axioms.

1. THE AXIOM OF SIMILARITY.

Either of two similar magnitudes is a substitute for the other, if only form be considered.

2. THE AXIOM OF EQUIVALENCY.

Either of two equivalent magnitudes is a substitute for the other, if only extent be considered.

3. THE AXIOM OF EQUALITY.

Either of two equal magnitudes is a substitute for the other, if both form and extent, or either, be considered.

23. Corollaries.

1. *If one of two similar magnitudes is similar to a third magnitude, the other is similar to that magnitude.*

For, if a, b, c, represent three magnitudes, and if (1) a is similar to b, and (2) b is similar to c, then, substituting a for b in (2), or c for b in (1), we have (3) a is similar to c.

2. *If one of two similar magnitudes is not similar to a third magnitude, the other is not similar to that magnitude.*

For, if (1) a is similar to b, and (2) b is not similar to c, then, substituting a for b in (2), we have (3) a is not similar to c.

3. *If one of two equivalent magnitudes is equivalent to a third magnitude, the other is equivalent to that magnitude.*

For, if (1) $a = b$, and (2) $b = c$, then, substituting a for b in (2), or c for b in (1), we have (3) $a = c$.

4. *If one of two equivalent magnitudes is greater than a third magnitude, the other is greater than that magnitude.*

For, if (1) $a = b$, and (2) $b > c$, then, substituting a for b in (2), we have (3) $a > c$.

5. *If one of two equivalent magnitudes is less than a third magnitude, the other is less than that magnitude.* (?)

6. *If equivalents be added to equivalents, the sums will be equivalent.*

For, let (1) $a = b$, (2) $c = d$; now (3) $a + c = a + c$, since the members are identical in extent; then, substituting b for a, and d for c in the second member of (3), we have (4) $a + c = b + d$.

7. *If equivalents be subtracted from equivalents, the remainders will be equivalent.* (?)

8. *If equivalents be multiplied by the same number, the products will be equivalent.* (?)

9. *If equivalents be divided by the same number, the quotients will be equivalent.* (?)

10. *Like powers of the numerical values of equivalents, referred to the same unit, are equivalent.* (?)

11. *Like roots of the numerical values of equivalents, referred to the same unit, are equivalent.* (?)

12. *If equivalents be added to inequivalents, the sums will be inequivalent, and the resulting inequivalency will subsist in the same sense.*

Let (1) $a = b$, (2) $c > d$. Let (3) $c = d + q$. Now (4) $a + d + q > a + d$ (**21**, 6). Then, substituting c for $d + q$ in the first member of (4), and b for a in the second member, we have (5) $a + c > b + d$.

13. *If equivalents be subtracted from inequivalents, the remainders will be inequivalent, and the resulting inequivalency will subsist in the same sense.* (?)

14. *If inequivalents be multiplied by the same number, the products will be inequivalent, and the resulting inequivalency will subsist in the same sense.* (?)

15. *If inequivalents be divided by the same number, the quotients will be inequivalent, and the resulting inequivalency will subsist in the same sense.* (?)

16. *If inequivalents be subtracted from equivalents, the remainders will be inequivalent, and the resulting inequivalency will subsist in a contrary sense.* (?)

17. *If equivalents be multiplied by unequal numbers, the products will be inequivalent, the greater product corresponding to the greater multiplier.* (?)

18. *If equivalents be divided by unequal numbers, the quotients will be inequivalent, the greater quotient corresponding to the less divisor.* (?)

19. *If two inequivalencies subsisting in the same sense be added, member to member, the resulting inequivalency will subsist in the same sense.*

Let $a > b$, and $c > d$. Let $a = b + p$, and $c = d + q$. Now, $b + p + d + q > b + d$ (**21,** 6). Then, substituting a for $b + p$, and c for $d + q$, we have $a + c > b + d$.

20. *If a magnitude is greater than the greater of two magnitudes, it is greater than the less.*

Let $a > b$, and $b > c$; then, $a + b > b + c$ (**23,** 19). Subtracting b from each member, we have $a > c$ (**23,** 13).

24. Scholiums.

1. Analogous corollaries can, in like manner, be deduced from the axiom of equality (**22,** 3).

2. The corollary from the axiom of equality, corresponding

to the first from the axiom of equivalency (**23**, 3), is usually stated as an axiom, thus: *Things equal to the same thing are equal to each other.*

25. Postulates.

1. THE POSTULATE OF POSITION.

A magnitude can have any position.

2. THE POSTULATE OF FORM.

A magnitude can have any form.

3. THE POSTULATE OF EXTENT.

A magnitude can have any extent.

26. Corollaries.

1. *A straight line can be drawn from any point, in any direction, to any extent.*

For, a magnitude can have any position and extent.

2. *A straight line can be drawn from any point to any other point.*

For, draw a line in any plane; move the plane, carrying with it the line, till the line passes through the first point; revolve the plane about the line as an axis, till it embraces the second point; revolve the line in the plane about the first point as a center, till it passes through the second point.

3. *A straight line passing through two fixed points is determined in position.*

For, if the line should change its position, it could no longer pass through the two points; hence, the two points determine the position of the line.

4. *Two straight lines having two points common are coincident.*

For, since a straight line passing through two points is determined in position, the two lines are identical in position, and hence coincident.

5. *Two straight lines can not intersect in more than one point.*

For, if they intersect in two points, they have two points common, and are therefore coincident.

6. *Two straight lines can not inclose a space.*

For, if they inclose a space, they would have two points common, and therefore be coincident, and the inclosed space would vanish.

7. *A finite straight line can be produced in either direction to any extent.*

For, a point can move from either extremity as origin, along the line to the other extremity, and forward, in the same direction indefinitely, thus tracing, beyond the extremity, the prolongation of the line to any extent.

8. *A finite straight line can be bisected.*

For, a point moving from either extremity of the line as origin, along the line toward the other extremity, will at each position divide the line into two parts. The part at first less than the other, will increase continuously, while the other will in like manner diminish, till, at length, the parts become equal, when the line is bisected.

9. *A plane angle can be bisected.*

For, let a straight line, drawn through the vertex in coincidence with one side of the angle, revolve about the vertex as a center, in the plane of the angle, from that side toward the other. The angle which the revolving line makes with the side left, at first less than the angle which it makes with the other side, increases continuously, while the other angle in like manner diminishes, till, at length, these angles become equal, when the angle is bisected.

BOOK I.

I. PERPENDICULAR AND OBLIQUE LINES.

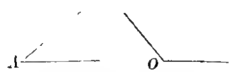
27. Definitions.

1. **Perpendicular lines** are lines which make a right angle with each other (**6,** 9).

Thus, if R is a right angle, its sides are perpendicular lines. If one line is perpendicular to another, the angle formed is a right angle, and the second line is perpendicular to the first.

2. **Oblique lines** are lines which make an oblique angle with each other (**6,** 10).

Thus, if A and O are oblique angles, their sides are oblique lines. If one line is oblique to another, the angle formed is an oblique angle, and the second line is oblique to the first.

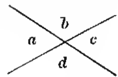


3. The **complement** of an angle is an angle which, added to the given angle, will make the sum a right angle.

4. The **supplement** of an angle is an angle which, added to the given angle, will make the sum two right angles.

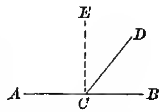
5. If two straight lines intersect each other, the two angles on the same side of one of the lines and on opposite sides of the other, are called *adjacent angles;* and the two angles on opposite sides of both lines are called *vertical angles.*

Thus, a and b, b and c, c and d, d and a, are adjacent angles; and a and c, b and d, are vertical angles.



28. Proposition I.—Theorem.

At any point in a straight line an indefinite number of perpendiculars to the line can be drawn.



Let C be a point of the line AB. From C, draw, in any plane embracing AB—for example, the plane of the paper—any line CD, making with AB the angles ACD, DCB, which must be equal or unequal. If these angles are equal, each will be a right angle (**6**, 9), and the line CD will be perpendicular to AB (**27**, 1). If these angles are unequal, let CD revolve, in the plane, about C as a center, so that the less angle shall continuously increase, and the greater in like manner diminish, till the two angles become equal, when each will be a right angle, and CD, in this position represented by CE, will be perpendicular to AB.

Now, let the plane revolve about AB as an axis, carrying with it the perpendicular CE. The angles, BCE, ECA, remaining right angles, the line CE will, in every position throughout the revolution, be perpendicular to AB. But CE takes an indefinite number of positions in the revolution, each of which is the position of a perpendicular to AB. Hence, an indefinite number of perpendiculars to AB can be drawn from the point C.

29. Corollaries.

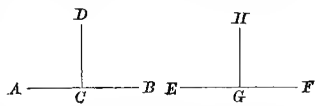
1. *From a point of a line only one perpendicular to the line can be drawn in a plane embracing the line.*

For, if the line revolve in the plane, either way, from the position in which the adjacent angles are equal, one of the angles would increase, the other diminish, and the angles, no longer equal, would not be right angles; hence, the line would not be a perpendicular.

The perpendicular produced across the line is a perpendicular on the other side of the line. (?)

2. *All right angles are equal.*

Let ACD, DCB, be right angles; also EGH, HGF. Then will these angles all be equal. For, place the figure ADB on EHF, so that AB shall coincide with EF, and C with G. Then CD will coincide with GH, otherwise there would be two perpendiculars to AB, drawn from C in a plane embracing AB, which is impossible.



Then, $ACD = EGH$, and $DCB = HGF$ (**21**, 5).
But, $ACD = DCB$, and $EGH = HGF$ (**6**, 9).
$\therefore$ $ACD = HGF$, and $DCB = EGH$ (**22**, 3).

3. *The complements of equal angles are equal.*

Let A, A', respectively, denote two angles; C, C', their respective complements; and R, a right angle.

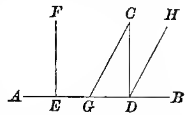
Then, $A + C = R$, and $A' + C' = R$ (**27**, 3).
$\therefore$ $A + C = A' + C'$ (**22**, 3; **24**, 2).
If $A = A'$, then $C = C'$ (**23**, 7).

4. *The supplements of equal angles are equal.* (?)

30. Proposition II.—Theorem.

From a point without a straight line, one perpendicular to the line can be drawn, and only one.

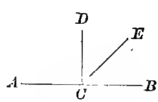
Let C be the point and AB the line. The point and the line are in the same plane, since any plane embracing AB can revolve about AB, as an axis, till it embraces the point C. At any point of AB, as E, let EF in the plane be perpendicular to AB, on the same side as C, forming the right angle AEF. Let this right angle move in the plane toward C, the side AE moving in AB, till FE passes through C and becomes CD. Since ADC is one position of the right angle AEF, CD is perpendicular to AB. Hence, from a point without a line, one perpendicular to the line can be drawn.



No other perpendicular to AB can be drawn from C; for, if CG is another perpendicular, AGC is a right angle. Let this angle move in the plane, AG moving in AB till G coincides with D, when GC, having moved, takes the position DH. Since AGC in its position ADH is a right angle, DH is perpendicular to AB. Then DC and DH are both perpendicular to AB in a plane embracing AB, which is impossible (**29**, 1). Hence, the supposition that a perpendicular differing from CD can be drawn from C to AB, which led to this impossibility, is false (**15**, 4). Therefore, from a point without a straight line, only one perpendicular to the line can be drawn.

31. Proposition III.—Theorem.

The sum of the two adjacent angles formed by one straight line meeting another, is equal to two right angles.



1. Let CD be perpendicular to AB, and let R denote a right angle.

Then, $ACD = R$, $DCB = R$ (**27**, 1).

$\therefore$ $ACD + DCB = 2R$ (**23**, 6).

2. Let CE be oblique to AB.

$$ACE + ECB = ACE - DCE + DCE + ECB,$$

since $-DCE$ and $+DCE$ cancel in the second member.

But, $ACE - DCE = ACD, \quad DCE + ECB = DCB.$

$\therefore \quad ACE + ECB = ACD + DCB.$

But, $ACD + DCB = 2R, \ \therefore \ ACE + ECB = 2R.$

32. Corollaries.

1. *If one of the two adjacent angles formed by one straight line meeting another is a right angle, the other is also a right angle.*

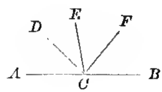
If not, the sum would not be equal to two right angles.

2. *The sum of all the consecutive angles formed by any number of straight lines, drawn from the same point of a straight line on the same side, is equal to two right angles.*

For, $ACF + FCB = 2R.$

But, $ACF = ACD + DCE + ECF.$

$\therefore \quad ACD + DCE + ECF + FCB = 2R.$

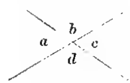


33. Proposition IV.—Theorem.

If two straight lines intersect, the vertical angles are equal.

For, $\quad a + b = 2R, \quad b + c = 2R$ (**31**).

$\therefore \quad a + b = b + c.$ (?) $\ \therefore \ a = c.$ (?)



Again, b is the supplement of a. (?)

Also, $\ d$ is the supplement of a, $\ \therefore \ b = d.$ (?)

34. Corollaries.

1. *If two straight lines intersect, the sum of the four angles formed is equal to four right angles.*

For, $\quad a + b = 2R$, and $c + d = 2R$ (**31**).

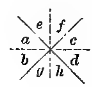
$\therefore \quad a + b + c + d = 4R$ (**23**, 6).

2. *The sum of all the angles formed by straight lines meeting at a common point is equal to four right angles.*

For, let two straight lines intersect at this point. The sum of the angles formed by the two lines will be equal to the sum of the angles formed by the given lines. But the sum of the angles formed by the two lines is equal to four right angles (**34,** 1). Hence, the sum of the angles formed by the given lines is equal to four right angles.

3. *The straight line bisecting an angle bisects its vertical angle.*

For, $a = d$, and $b = c$ (**33**).
If $a = b$, then $a = c$, $\therefore$ $d = c$. (?)

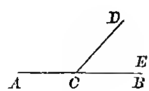


4. *The bisectors of the two pairs of vertical angles are perpendicular to each other.*

For, $e = f$. But, $a = b$, and $b = c$, $\therefore$ $a = c$.
$\therefore$ $e + a = f + c$, $\therefore$ the bisectors are perpendicular.

35. Proposition V.—Theorem.

If the sum of two adjacent angles is equal to two right angles, their exterior sides are in the same straight line.



Let $ACD + DCB = 2R$. Then ACB is a straight line.

Let AC be produced to E.

Then, $ACD + DCE = 2R$. (?)
$\therefore$ $ACD + DCB = ACD + DCE$.

$\therefore$ $DCB = DCE$. (?) $\therefore$ CE coincides with CB; otherwise, DCB and DCE would not be equal.

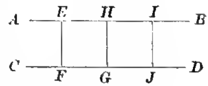
But ACE is a straight line; $\therefore$ ACB is a straight line.

II. PARALLEL LINES.

36. Definitions.

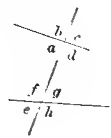
1. **Parallel lines** are lines every-where equally distant.

Thus, if AB and CD are parallel, the perpendicular distance from E to CD is equal to that from G to AB, or to that from I to CD, and so on.



2. A **secant** of two lines is a line intersecting both; and,

1st. **Interior angles** are those which lie within the two lines. Thus, a, d, g, f.



2d. **Exterior angles** are those which lie without the two lines. Thus, b, c, h, e.

3d. **Interior angles on the same side** are those which lie within the two lines, on the same side of the secant. Thus, a and f, d and g.

4th. **Exterior angles on the same side** are those which lie without the two lines, on the same side of the secant. Thus, b and e, c and h.

5th. **Alternate interior angles** are those which lie within the two lines, on opposite sides of the secant, but not adjacent. Thus, a and g, d and f.

6th. **Alternate exterior angles** are those which lie without the two lines, on opposite sides of the secant, but not adjacent. Thus, b and h, c and e.

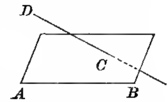
7th. **Corresponding angles** are those which are similarly situated with respect to the two lines and the secant. Thus, b and f, a and e, c and g, d and h.

It will be observed that the two lines are not necessarily parallel.

37. Proposition VI.—Theorem.

Two parallel lines lie in the same plane.

Let AB be one of the parallels, and C one point of the other. Conceive a plane embracing AB to revolve about AB as an axis till it shall embrace the point C. Then, if the other parallel does not lie in this plane, it will pierce the plane at C, the point common to the line and plane, and diverge from the plane on each side of C.



Since the line and plane are divergent and do not change their directions, we shall find, on passing from C along this line, points farther and farther from the plane, till, at length, we shall find a point D, whose shortest distance from the plane is greater than the shortest distance of C from AB; and since AB, the axis about which the plane revolved, is a line of the plane, the shortest distance of D from AB is greater than the shortest distance of C from AB; that is, the lines are not every-where equally distant, and hence not parallel, which is contrary to the hypothesis.

Since the supposition that the parallel to AB through C does not lie in the plane embracing AB and C leads to a contradiction, this supposition is false; hence, this parallel must lie in this plane. Therefore, the two parallels lie in the same plane.

38. Proposition VII.—Theorem.

Parallel lines can not meet, however far either way both be produced.

For, if they could meet, they would not be every-where equally distant, and hence not parallel, which is contrary to the hypothesis; therefore they can not meet.

39. Proposition VIII.—Theorem.

Lines not parallel and lying in the same plane must meet if sufficiently produced.

For, if not parallel, they are not every-where equally distant; and, since they lie in the same plane, must approach when produced one way or the other; and, since straight lines continue in the same direction, must continue to approach if produced farther; and if sufficiently produced, must meet.

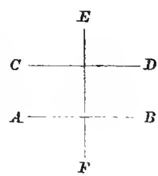
40. Proposition IX.—Theorem.

Lines which, lying in the same plane, can not meet are parallel.

For, if not parallel, they must meet if sufficiently produced, which is contrary to the hypothesis; hence they are parallel.

41. Proposition X.—Theorem.

Two straight lines lying in the same plane and respectively perpendicular to a third straight line lying in that plane, are parallel.

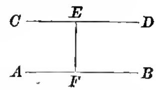


Let AB and CD be perpendicular to EF, the three lines lying in the same plane; then will AB and CD be parallel. For, if AB and CD, lying in the same plane, are not parallel, they will meet if sufficiently produced. Then we shall have two perpendiculars drawn from the same point—their point of meeting—to the same straight line, which is impossible (**30**). Hence they can not meet (**15**, 4). Since AB and CD lie in the same plane and can not meet, they are parallel (**40**).

42. Corollaries.

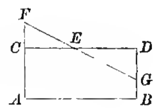
1. *Through any point a line can be drawn parallel to a given line.*

Let E be the point and AB the line. From E one perpendicular can be drawn to AB, and only one. (?) Let EF be this perpendicular. Then, in the plane of AB and EF, one perpendicular to EF can be drawn through E, and only one (**29,** 1). Let CD be this perpendicular. Then AB and CD, lying in the same plane and perpendicular to a third line, EF, in that plane, are parallel (**41**).



2. *Through the same point only one parallel can be drawn to the same line.*

Let E be the point and AB the line. By Cor. 1, one parallel to AB can be drawn through E. Let CD be this parallel. If possible, let FG, not coincident with CD, be drawn through E, parallel to AB. Since parallels lie in the same plane, these lines all lie in the plane embracing AB and E.

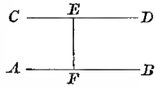


Let FA and DB be perpendicular to AB. Now, $FA > CA$ (**21,** 6); but, $CA = DB$, since AB and CD are parallel (**36,** 1). Therefore, $FA > DB$ (**22,** 3). But, $DB > GB$, $\therefore$ $FA > GB$ (**23,** 20). Hence, the lines AB and FG are not every-where equally distant, and therefore are not parallel.

3. *A straight line perpendicular to one of two parallels and lying in the same plane is perpendicular to the other.*

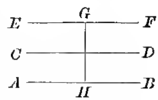
Let AB and CD be parallel, and EF perpendicular to AB; then will EF be perpendicular to CD. For, in the

plane of AB and EF, let a perpendicular to EF be drawn at E. Then this perpendicular will be parallel to AB (**41**), and must, therefore, coincide with CD; otherwise, there would be two parallels to AB drawn through the point E, which is impossible (Cor. 2). Since CD coincides with the perpendicular to EF through E, CD is itself a perpendicular; hence EF, which is perpendicular to AB, is also perpendicular to CD, which is parallel to AB.



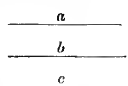
4. *Two straight lines lying in the same plane and respectively parallel to a third straight line lying in that plane are parallel to each other.*

Let CD and EF, lying in the same plane, be respectively parallel to AB, lying in that plane. Then will CD and EF be parallel. For, let GH, lying in the same plane, be perpendicular to AB; then GH will also be perpendicular to CD and to EF (**42**, 3). Hence, CD and EF, lying in the same plane and perpendicular to a third line, GH, lying in that plane, are parallel. (?)



5. *If one of three parallels lying in the same plane move in that plane toward the second, but in such a manner as to continue parallel to the third, it will, in one of its positions, coincide with the second.*

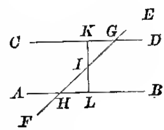
Let a, b, c, be such parallels, and let a move in the stated manner toward b, till it meets b in one point. Then, if a and b do not coincide, they will intersect and have but one point common. We should then have two lines through the same point parallel to the same line, which is impossible (**42**, 2); hence a and b coincide.



43. Proposition XI.—Theorem.

If a straight line intersect two straight lines lying in the same plane, making the sum of the interior angles on the same side of the secant equal to two right angles, the two lines are parallel.

Let EF intersect AB and CD, lying in the same plane, making $EHB + FGD =$ two right angles. Then will AB and CD be parallel. For, through I, the middle point of GH, draw KL perpendicular to AB. The vertical angles, KIG, HIL, are equal (**33**). By hypothesis,



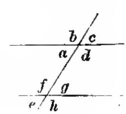
$$IHL + FGD = 2R; \text{ but } IGK + FGD = 2R \ (\mathbf{31}).$$
$$\therefore\ IHL + FGD = IGK + FGD; \ \therefore IHL = IGK \ (\mathbf{23}, 7).$$

Now, conceive the portion of the figure below I to revolve, in the same plane, about I as a center, till IH coincides with IG, and H with G. Since $KIG = HIL$, IL will fall in IK; hence, L will fall in IK. Since $IHL = IGK$, HL will fall in GC; hence, L will fall in GC. Since L is in both IK and GC, it must be at their intersection, K. Then ILH and IKG coincide, and hence are equal. But ILH is a right angle by construction; therefore IKG is a right angle. Hence, AB and CD, both perpendicular to KL and lying in the same plane, are parallel (**41**).

44. Corollaries.

1. *If the sum of the exterior angles on the same side is equal to two right angles, the two lines are parallel.*

Let $b + e =$ two right angles. But, $b = d$, and $e = g$ (**33**). $\therefore$ $d + g =$ two right angles; therefore the two lines are parallel (**43**).



2. *If either two alternate interior angles are equal, the two lines are parallel.*

Let $a = g$; then $a + d = g + d$. (?) But $a + d = 2R$. (?) $\therefore$ $g + d = 2R$; therefore the two lines are parallel (**43**).

3. *If either two alternate exterior angles are equal, the two lines are parallel.*

Let $c = e$; but $c = a$, and $e = g$; (?) $\therefore$ $a = g$; (?) therefore the two lines are parallel. (?)

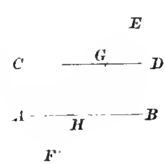
4. *If any two corresponding angles are equal, the lines are parallel.*

Let $e = g$; but $e = a$; $\therefore$ $a = g$; therefore the lines are parallel. (?)

Remark.—Let the student prove these four corollaries by taking, in each case, two angles other than those taken above.

45. Proposition XII.—Theorem.

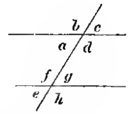
If a straight line intersect two parallel straight lines, the sum of the interior angles on the same side of the secant is equal to two right angles.



Let EF intersect the parallels AB and CD. The line through G in the plane of AB and G which makes the sum of the interior angles on the same side equal to two right angles, is parallel to AB (**43**), and must, therefore, coincide with CD; otherwise we should have two different lines through the same point parallel to the same straight line, which is impossible (**42**, 2). Hence, $GHB + HGD =$ two right angles.

46. Corollaries.

1. *If a straight line intersect two parallel straight lines, the sum of the exterior angles on the same side is equal to two right angles.*



For, $d + g =$ two right angles (**45**).
But, $d = b$, and $g = e$. (?)
$\therefore$ $b + e =$ two right angles (**22**, 3).

2. *If a straight line intersect two parallel straight lines, the alternate interior angles are equal.*

For, $a + d = 2R$ (**31**), and $d + g = 2R$ (**45**).
$\therefore$ $a + d = d + g$; (?) $\therefore$ $a = g$. (?)

3. *If a straight line intersect two parallel straight lines, the alternate exterior angles are equal.*

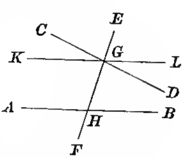
For, $a = g$; (?) but, $a = c$, and $g = e$; (?) $\therefore$ $c = e$. (?)

4. *If a straight line intersect two parallel straight lines, any two corresponding angles are equal.*

For, $a = g$; (?) but, $a = c$; (?) $\therefore$ $c = g$. (?)

47. Proposition XIII.—Theorem.

If a straight line intersect two other straight lines lying in the same plane, making the sum of the interior angles on either side of the secant less than two right angles, the two lines will meet on that side of the secant if sufficiently produced.



Let EF intersect AB and CD, making $GHB + HGD <$ two right angles. Then will AB and CD meet if sufficiently produced. For, if AB and CD lying in the same plane can not meet, they are parallel (**40**); $\therefore$ $GHB + HGD = 2R$ (**45**);

which is contrary to the hypothesis. Hence, AB and CD will meet if sufficiently produced.

To find on which side of the secant the lines will meet, draw KL through G, parallel to AB. Then HGL is the supplement of GHB; (?) but, by hypothesis, HGD is less than the supplement of GHB; therefore, $HGD < HGL$. Hence, GD lies within the parallels; hence, GC lies without. Therefore, CD can not meet AB on the same side of the secant as C, though it may meet it on the same side as D. But CD must meet AB if sufficiently produced; hence, CD must meet AB on the same side of the secant as D; that is, on that side on which the sum of the interior angles is less than two right angles.

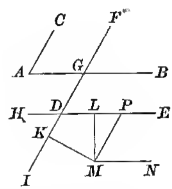
48. Proposition XIV.—Theorem.

Two angles are equal in the following cases:

1. *If their sides are respectively parallel and lie in the same direction from their vertices as origins.*

2. *If their sides are respectively parallel and lie in opposite directions from their vertices as origins.*

3. *If the sides of one respectively meet at right angles the sides of a supplemental angle of the other.*



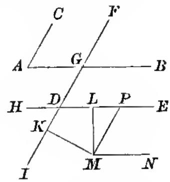
1. Let the sides of BAC and EDF be respectively parallel and lie in the same direction. Then, $BAC = EDF$.

For, $BAC = BGF$ (**46**, 4);
also, $EDF = BGF$; $\therefore$ $BAC = EDF$.

2. Let the sides of BAC and HDI be respectively parallel and lie in opposite directions. Then, $BAC = HDI$.

For, $BAC = AGD$ (**46,** 2); $HDI = AGD$, (?)
$\therefore$ $BAC = HDI$.

3. Let the sides of KML respectively meet at right angles the sides of IDE, the supplement of EDF. Then, $EDF = KML$. For, draw MN, MP, respectively parallel to DE, DF, and in the same direction; then, $EDF = NMP$ (**48,** 1), MN is perpendicular to ML, and MP to MK (**42,** 3). But $KML = NMP$ (**29,** 3); $\therefore$ $EDF = KML$. (?)



Also, EDF is equal to any angle whose sides are respectively parallel to the sides of KML, and lie in the same direction or in opposite directions. (?)

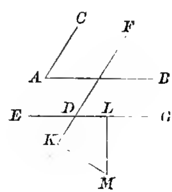
49. Proposition XV.—Theorem.

Two angles are supplemental in the following cases:

1. *If two of their sides are parallel and lie in the same direction, and the other two sides are parallel and lie in opposite directions.*

2. *If the sides of the one respectively meet at right angles the sides of the other.*

1. Let AC and DF be parallel and lie in the same direction, and AB and DE be parallel and lie in opposite directions. Then, EDF is the supplement of BAC.



For, $BAC = GDF$ (**48,** 1); but EDF is the supplement of GDF; (?) therefore, EDF is the supplement of BAC. (?)

2. Let the sides of KDG and KML respectively meet at right angles. Then KML is the supplement of KDG. For,

$KML = GDF$ (**48,** 3); but GDF is the supplement of KDG; therefore KML is the supplement of KDG.

Also, KML is the supplement of any angle whose sides are respectively parallel to the sides of KDG, and lie in the same direction or in opposite directions. (?)

50. Exercises.

1. What is the locus (**7**) of all the points in a plane embracing a line, which are equally distant from that line?

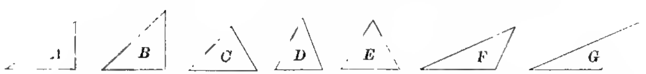
2. Of how many lines does the locus of Ex. 1 consist?

3. Does this locus embrace all the points equally distant from the line? What then is the locus of all the points equally distant from the line?

III. TRIANGLES.

51. Definitions and Classification.

1. A **triangle** is a polygon of three sides (**4,** 6).



2. A **right triangle** is a triangle having a right angle.

3. An **oblique triangle** is a triangle having all its angles oblique.

Thus, A, B, are right triangles; C, D, E, F, G, are oblique.

4. An **acute triangle** is an oblique triangle having all its angles acute.

5. An **obtuse triangle** is an oblique triangle having an obtuse angle.

Thus, C, D, E, are acute triangles; F, G, are obtuse.

6. A **scalene triangle** is a triangle having no two sides equal.

7. An **isosceles triangle** is a triangle having at least two sides equal.

Thus, *A*, *C*, *F*, are scalene triangles; *B*, *D*, *E*, *G*, are isosceles.

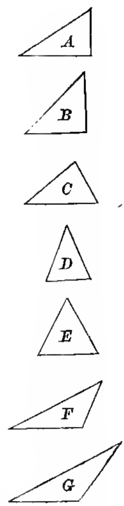
8. A **bi-equilateral triangle** is an isosceles triangle having only two equal sides. It is usually called *isosceles*.

9. A **tri-equilateral triangle** is an isosceles triangle having its three sides equal. It is usually called *equilateral*.

Thus, *B*, *D*, *G*, are bi-equilateral triangles; *E* is tri-equilateral.

10. Summary classification of triangles:

TRIANGLES
- Right
  - Scalene
  - Isosceles—Bi-equilateral
- Oblique
  - Acute
    - Scalene
    - Isosceles
      - Bi-equilateral
      - Tri-equilateral
  - Obtuse
    - Scalene
    - Isosceles—Bi-equilateral



11. The **hypotenuse** of a right triangle is the side opposite the right angle.

12. The **base** of a triangle is the side on which it is assumed to stand.

13. In a bi-equilateral triangle, the side unequal to either of the others is generally regarded as the base.

14. The **vertical angle** of a triangle is the angle opposite the base.

15. The **vertex** of a triangle is the vertex of the vertical angle.

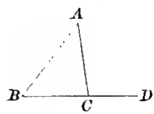
16. The **altitude** of a triangle is the perpendicular from the vertex to the base.

17. A **medial line** of a triangle is the line drawn from any vertex to the middle point of the opposite side.

18. An **exterior angle** is the angle which the prolongation of any side of the triangle makes with another side.

19. The **adjacent angle** of an exterior angle is the angle one of whose sides is prolonged.

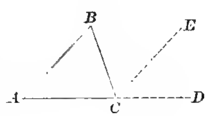
20. **Opposite interior angles** are the two angles not adjacent to the exterior angle.



Thus, ACD is an exterior angle; ACB is the adjacent angle of ACD; A and B are the opposite interior angles.

52. Proposition XVI.—Theorem.

The sum of the three angles of any triangle is equal to two right angles.



Let ABC be a triangle. Then the sum of its angles is equal to two right angles. For, producing any side as AC and drawing CE parallel to AB and on the same side of AC, we have

$$ACB + BCE + ECD = \text{two right angles } (\mathbf{32},\ 2).$$
$$BCE = ABC\ (\mathbf{46},\ 2), \quad ECD = BAC\ (\mathbf{46},\ 4).$$
$$\therefore \quad ACB + ABC + BAC = \text{two right angles}.$$

53. Corollaries.

1. *An exterior angle is equal to the sum of the opposite interior angles.*

For, $ACB + BCD = BAC + ABC + ACB$ (**31**), (**52**).

$\therefore \quad BCD = BAC + ABC$ (**23**, 7).

2. *An exterior angle is greater than either opposite interior angle.* (?)

3. *If one angle of a triangle is right, each of the others is acute.* (?)

4. *If one angle of a triangle is obtuse, each of the others is acute.* (?)

5. *In a right triangle, the sum of the two acute angles is equal to a right angle, and each acute angle is the complement of the other.* (?)

6. *In an acute triangle, the sum of any two angles is greater than a right angle.* (?)

7. *In an obtuse triangle, the sum of the two acute angles is less than a right angle.* (?)

8. *In any triangle, any angle is the supplement of the sum of the other two.* (?)

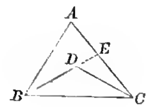
9. *If the sum of two angles of one triangle is equal to the sum of two angles of another, the third angle of the one is equal to the third angle of the other.* (?)

10. *Each angle of an equiangular triangle is one-third of two right angles or two-thirds of one right angle.* (?)

54. Proposition XVII.—Theorem.

The angle contained by two straight lines drawn from any point within a triangle to the extremities of one of the sides is greater than the angle contained by the other sides of the triangle.

Let DB and DC be straight lines drawn from any point D within the triangle ABC to the extremities of BC. Then, the angle BDC is greater than the angle BAC. For,



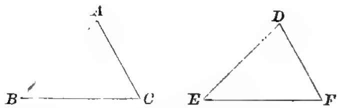
$$BDC > BEC, \quad BEC > BAC \text{ (53, 2)}.$$

Hence, $BDC > BAC$ (**23**, 20).

55. Proposition XVIII.—Theorem.

Two triangles are equal if two sides and the included angle of the one are respectively equal to two sides and the included angle of the other.

Let ABC and DEF be two triangles, having the side AB equal to the side DE, the side AC equal to



the side DF, and the included angle A equal to the included angle D. Then the triangle ABC is equal to the triangle DEF. For, let ABC be placed upon DEF, so that the angle A shall coincide with the equal angle D; then, the side AB will coincide with the equal side DE, and the side AC with the equal side DF, the point B with E, C with F, and BC with EF (**26**, 4). Hence, the triangles coincide, and are therefore equal (**21**, 5).

56. Corollaries.

1. *If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another, the third sides will be equal, and the remaining angles respectively equal.*

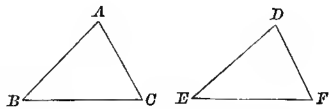
For, since the triangles coincide, the like parts coincide, and hence are equal.

2. *If two triangles are equal, the equal sides are opposite equal angles, and the equal angles opposite equal sides.*

3. *If the equal sides of the equal angles of two triangles are reversed in position, the triangles can be made to coincide, if one be turned over.*

57. Proposition XIX.—Theorem.

Two triangles are equal if a side and the adjacent angles of the one are respectively equal to a side and the adjacent angles of the other.



Let ABC and DEF be two triangles, having the side BC equal to the side EF, the angle B equal to the angle E, and the angle C equal to the angle F. Then will the triangles be equal. For, let ABC be placed upon DEF, so that BC shall coincide with its equal EF; then, since the angle B is equal to the angle E, BA will fall on ED, and, since the angle C is equal to the angle F, CA will fall on FD; hence, A, the intersection of BA and CA, will coincide with D, the intersection of ED and FD. Hence, the triangles coincide, and are therefore equal.

58. Corollary.

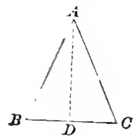
If a side and the adjacent angles of one triangle are respectively equal to a side and the adjacent angles of another, the third angles are equal, and the remaining sides respectively equal—the equal sides lying opposite equal angles.

For, since the triangles can be made to coincide, the corresponding parts are equal.

59. Proposition XX.—Theorem.

The angles opposite the equal sides of an isosceles triangle are equal.

Let ABC be an isosceles triangle, having the side AC equal to the side AB. Then is the angle B, opposite the side AC, equal to the angle C, opposite the equal side AB.



Let AD bisect the angle BAC (**26**, 9). The triangles ABD and ACD are equal, since AD is common, AB and AC are equal by hypothesis, and the angles BAD and CAD equal (**55**). Hence, the angles B and C are equal, since they are opposite the common side AD (**56**, 2). But B and C are opposite the equal sides, AC, AB, of the isosceles triangle ABC. Hence, in an isosceles triangle, the angles opposite the equal sides are equal.

60. Corollaries.

1. *The line bisecting the vertical angle of an isosceles triangle bisects the base at right angles.* (?)

2. *The line bisecting the base of an isosceles triangle at right angles bisects the vertical angle.* (?)

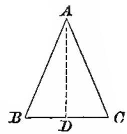
3. *The line joining the vertex of an isosceles triangle and the middle point of the base bisects the vertical angle, and is perpendicular to the base.* (?)

4. *Every equilateral triangle is equiangular.* (?)

61. Proposition XXI.—Theorem.

If two angles of a triangle are equal, the sides opposite are equal.

Let ABC be a triangle, having the angle B equal to the angle C; then is the side AC, opposite B, equal to the side AB, opposite C.



Let AD bisect the angle BAC (**26**, 9). Then, in the triangles, ABD, ACD, $B = C$, and $BAD = CAD$; $\therefore$ $ADB = ADC$. Hence, the triangles, ABD, ACD, are equal (**57**); therefore, $AB = AC$. (?)

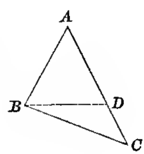
62. Corollary.

Every equiangular triangle is equilateral. (?)

63. Proposition XXII.—Theorem.

The angle opposite the greater of two unequal sides of a triangle is greater than the angle opposite the less.

Let the side AC of the triangle ABC be greater than the side AB. Then, the angle ABC, opposite the side AC, is greater than the angle ACB, opposite the side AB.

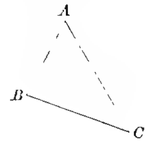


On AC cut off AD equal to AB, and draw BD. Then, the angle ABD is equal to the angle ADB (**59**); and the angle ADB is greater than the angle DCB (**53**, 2); hence, the angle ABD is greater than the angle ACB. But the angle ABC is greater than the angle ABD (**21**, 6); hence, the angle ABC is greater than the angle ACB.

64. Proposition XXIII.—Theorem.

The side opposite the greater of two unequal angles of a triangle is greater than the side opposite the less.

Let the angle B of the triangle ABC be greater than the angle C; then, the side AC, opposite B, is greater than the side AB, opposite C. Now, either $AC = AB$, $AC < AB$, or $AC > AB$. If $AC = AB$, $B = C$ (**59**), which is contrary to the hypothesis; therefore, AC is not equal to AB (**15**, 4). If $AC < AB$, $B < C$ (**63**), which is contrary to the hypothesis; $\therefore$ AC is not less than AB. Now, neither $AC = AB$, nor $AC < AB$; $\therefore$ $AC > AB$.

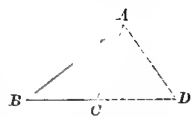


65. Proposition XXIV.—Theorem.

Any side of a triangle is less than the sum of the other sides.

Let ABC be a triangle. Produce BC till $CD = CA$, and draw AD.

Since $CD = CA$, the angle CAD is equal to the angle CDA (**59**). The angle BAD is greater than the angle CAD; hence, the angle BAD is greater than the angle BDA. Therefore, the side BD is greater than the side AB (**64**); $\therefore$ $AB < BD$.



$BD = BC + CD$; but, $CD = CA$; $\therefore$ $BD = BC + CA$.

But, $\quad AB < BD$; $\therefore$ $AB < BC + CA$.

Likewise, we find

$$BC < CA + AB, \text{ and } CA < AB + BC.$$

66. Corollary.

Any side of a triangle is greater than the difference of the other sides.

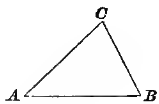
Denoting the sides respectively by a, b, c, we have

$$a + b > c; \therefore a > c - b, \quad b > c - a.$$
$$b + c > a; \therefore b > a - c, \quad c > a - b.$$
$$c + a > b; \therefore c > b - a, \quad a > b - c.$$

67. Proposition XXV.—Theorem.

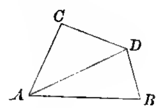
The shortest distance from one point to another is the straight line having these points for its extremities.

1. Let A and B be the points, AB a straight line having these points for its extremities, ACB a broken line having two components and the same extremities.



ACB is a triangle; $\therefore AB < AC + CB$ (65).

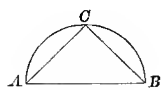
2. Let AB be a straight line, $ACDB$ a broken line having more than two components and the same extremities. Then,



$$AB < AC + CD + DB. \quad \text{For,}$$

$AD < AC + CD$; (?) $\therefore AD + DB < AC + CD + DB$ (?)
$AB < AD + DB$; (?) $\therefore AB < AC + CD + DB$ (?)

3. Let AB be a straight line, ACB a curve having the same extremities. Then



$$AB < \text{the curve.}$$

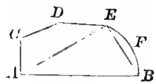
For, drawing the straight lines, AC, CB,

$$AB < AC + CB.$$

Draw a straight line from A to the middle point of the curve AC, and from this point to C, and so on round to B, thus forming a new broken line of double the number of components, which can be proved, as above, greater than $AC + CB$.

In like manner, successive broken lines can be formed, each of which will exceed the preceding and more nearly coincide with the curve, which is therefore greater than any of them. But, $AB < AC + CB$; $\therefore$ $AB <$ the curve. (?)

4. Let AB be a straight line, $ACDEFB$ a mixed line having the same extremities. Then,



$$AB < AC + CD + DE + EFB.$$

For, $AE < AC + CD + DE$, $\quad EB < EFB$;

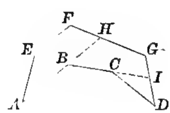
$\therefore \quad AE + EB < AC + CD + DE + EFB$. (?)

But, $AB < AE + EB$; $\therefore$ $AB < AC + CD + DE + EFB$.

68. Proposition XXVI.—Theorem.

A convex broken line is less than any enveloping line having the same extremities.

1. Let $ABCD$ be a convex broken line, $AEFGD$ an enveloping broken line having the same extremities. Then,



$$ABCD < AEFGD.$$

Produce $\quad AB$ to H, and BC to I.

Then, $\quad AB + BH < AE + EF + FH$ (67, 2).

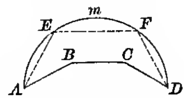
Also, $\quad BC + CI < BH + HG + GI$,

and $\quad CD < CI + ID$ (65).

Adding these inequations, subtracting BH and CI from both members, and substituting FG for $FH + HG$ and GD for $GI + ID$, we have

$$AB + BC + CD < AE + EF + FG + GD.$$

2. Let $ABCD$ be a convex broken line and AmD an enveloping curve. Inscribe in the curve the broken line $AEFD$. Then,



$$ABCD < AEFD \quad (\mathbf{68},\ 1).$$

But, $AEFD < AmD$ (**67**, 3); $\therefore\ ABCD < AmD.$

69. Proposition XXVII.—Theorem.

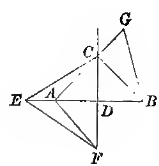
If from a point without a straight line a perpendicular be drawn to this line and oblique lines be drawn from the same point to different points of the line, then:

1. *The perpendicular is shorter than any oblique line.*

2. *Two oblique lines which meet the given line at equal distances from the foot of the perpendicular are equal.*

3. *Of two oblique lines which meet the given line at unequal distances from the foot of the perpendicular, that which cuts off the greater distance is the greater.*

4. *Of two oblique lines drawn from the same point without the perpendicular, in the plane of the perpendicular and line, cutting off on the line equal distances from the foot of the perpendicular, that which intersects the perpendicular is the greater.*



1. Let CD be perpendicular to AB, CA an oblique line. Then, $CD < CA$. Produce CD till $DF = CD$, and draw AF. The angle ADC is a right angle (**27**, 1); therefore the angle ADF is a right angle (**32**, 1). The triangles, ADC, ADF, are equal since AD is common, $DC = DF$ by construction, and the included angle $ADC =$ the included angle ADF (**29**, 2) (**55**). Then, AC opposite the angle ADC, is equal to AF

opposite the equal angle ADF. Since $CD = DF$ and $CA = AF$, $CD = \frac{1}{2}CF$ and $CA = \frac{1}{2}(CA + AF)$. Since ACF is a triangle,

$$CF < CA + AF; \quad \therefore \quad \tfrac{1}{2}CF < \tfrac{1}{2}(CA + AF) \quad \textbf{(23, 15)}.$$

$$\therefore \quad CD < CA \quad \textbf{(22, 3)}.$$

2. Let $DA = DB$. Then the triangles, ACD, BCD, are equal; for, CD is common, $DA = DB$ by hypothesis, and the included angles CDA and CDB are equal. Hence, CA opposite the angle CDA, is equal to CB opposite the equal angle CDB.

3. Let $DE > DA$, and draw EF. Then, the triangles, ECD, EFD, are equal. (?) $\therefore$ $CE = EF$. (?)

$$\therefore \quad CE = \tfrac{1}{2}(CE + EF).$$

But, $CA = \frac{1}{2}(CA + AF)$, and $CE + EF > CA + AF$ **(68)**.

$\therefore \quad \frac{1}{2}(CE + EF) > \frac{1}{2}(CA + AF)$, $\therefore$ $CE > CA$.

4. Let G be a point in the plane CDB. Draw GB, and GA intersecting the perpendicular in C. Then we have

$$GC + CB > GB. \quad \text{But,} \quad CA = CB.$$

$$\therefore \quad GC + CA > GB, \quad \therefore \quad GA > GB.$$

70. Corollaries.

1. *From a given point to a given straight line only two equal straight lines can be drawn.* (?)

2. *The perpendicular intersecting a straight line at its middle point is the locus of all the points in the plane of the line and perpendicular which are equally distant from the extremities of the line.* (?)

3. *If each of two points of one straight line is equally distant from the extremities of another straight line in the same plane, the first line is perpendicular to the second at its middle point.* (?)

4. *If the plane embracing a straight line revolve about this line as an axis, carrying with it the perpendicular at the middle point, this perpendicular indefinitely extended will generate a plane of indefinite extent which will be the general locus of all the points equally distant from the extremities of the line.* (?)

71. Scholium.

By the distance from a point to a line, we are to understand the perpendicular or shortest distance.

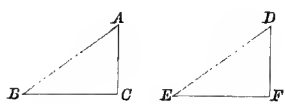
72. Proposition XXVIII.—Theorem.

Two right triangles are equal in the following cases:

1. *If the hypotenuse and a side of the one are respectively equal to the hypotenuse and a side of the other.*

2. *If the hypotenuse and an adjacent angle of the one are respectively equal to the hypotenuse and an adjacent angle of the other.*

3. *If a side adjacent to the right angle, and the adjacent or opposite acute angle of the one, are respectively equal to a side adjacent to the right angle, and the adjacent or opposite acute angle of the other.*



1. Let ABC, DEF, be right triangles having the hypotenuses, AB, DE, equal, and the sides, AC, DF, equal. Then the triangles are equal.

For, place the triangle ABC on DEF so that the side AC shall coincide with its equal DF; then, since the right angles are equal, CB will fall on FE, and the point B will be found somewhere in FE, either at E or at the right or left of E.

B can not fall at the right of E; for then AB would be less than DE (**69**, 3), which is contrary to the hypothesis. Neither can B fall at the left of E, for then AB would be greater than DE, which is contrary to the hypothesis. Hence B must fall at E; in which case, the triangles coincide, and are therefore equal.

2. Let the hypotenuses, AB, DE, be equal, also the adjacent angles, A, D. Then will the triangles be equal. For, B is the complement of A and E of D (**53**, 5); but $A = D$, $\therefore$ $B = E$ (**29**, 3). Hence, the triangles having the side AB and the adjacent angles, A, B, of the one, respectively equal to the side DE and the adjacent angles, D, E, of the other, are equal (**57**).

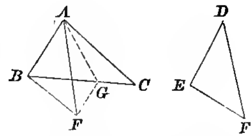
3. If $BC = EF$ and $B = E$, then, since $C = F$ (**29**, 2), the triangles are equal (**57**). If $BC = EF$ and $A = D$, then $B = E$ (**53**, 5), and the triangles are equal, as just shown.

73. Proposition XXIX.—Theorem.

If two triangles have two sides of the one respectively equal to two sides of the other, and the included angle of the one greater than the included angle of the other, the third side of the one having the greater included angle is greater than the third side of the other.

Let ABC and DEF be two triangles having $AB = DE$,

$AC = DF$, and the angle $BAC >$ the angle EDF. Then the side $BC >$ the side EF.



For, place DEF on BAC so that it shall take the position ABF, bisect the angle CAF by AG, and draw GF.

In the triangles, CAG, FAG, we have $AC = AF$ by hypothesis, AG common, and the angle $CAG = FAG$ by construction; $\therefore$ $GC = GF$. (?)

Now, $\quad BG + GF > BF$; but, $GC = GF$,

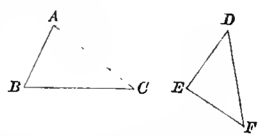
$\therefore \quad BG + GC > BF$.

But, $\quad BG + GC = BC$, and $BF = EF$,

$\therefore \quad BC > EF$.

74. Proposition XXX.—Theorem.

If two triangles have two sides of the one respectively equal to two sides of the other and the third side of the one greater than the third side of the other, the angle opposite the greater third side is greater than the angle opposite the less.



If $AB = DE$, $AC = DF$, and $BC > EF$, then $A > D$. For, either $A = D$, $A < D$, or $A > D$.

If $A = D$, $BC = EF$ (**56**, 1).

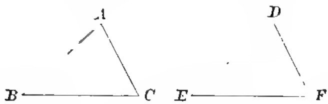
If $A < D$, $BC < EF$ (**73**).

Both these results are contrary to the hypothesis that $BC > EF$; hence, neither $A = D$, nor $A < D$ (**15**, 4); $\therefore$ $A > D$.

75. Proposition XXXI.—Theorem.

Triangles mutually equilateral are mutually equiangular and equal.

Let ABC and DEF be two triangles having $AB = DE$, $AC = DF$, and $BC = EF$.



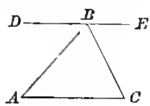
Then, $A = D$, $B = E$, $C = F$. For, either $A > D$, $A < D$, or $A = D$.

If $A > D$, $BC > EF$; if $A < D$, $BC < EF$.?

Both these results are contrary to the hypothesis that $BC = EF$; hence, neither $A > D$, nor $A < D$; $\therefore$ $A = D$. In like manner, it is proved that $B = E$, and $C = F$. Hence, the triangles can be made to coincide, and are therefore equal.

76. Exercises.

1. Prove by means of the annexed diagram, in which DE is parallel to AC, that the sum of the three angles of a triangle is equal to two right angles.

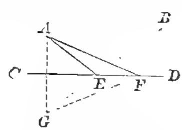


2. The sum of the three straight lines drawn from any point within a triangle to the vertices is greater than one-half the sum of the three sides (**65**).

3. The sum of the three straight lines drawn from any point within a triangle to the vertices is less than the sum of the three sides (**68**).

4. In a right triangle, the middle point of the hypotenuse is equally distant from the vertices of the three angles (**61**).

5. If $AEC = BED$, and F is any other point of CD than E, prove that $AE + EB < AF + FB$.

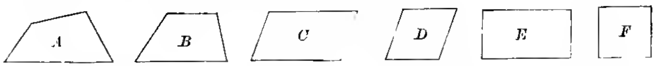


6. If one of the acute angles of a right triangle is double the other, the hypotenuse is double the shortest side.

IV. QUADRILATERALS.

77. Definitions and Classification.

1. A **quadrilateral** is a polygon of four sides. Thus, *A*, *B*, *C*, *D*, *E*, *F*, are quadrilaterals.



2. A **trapezium** is a quadrilateral having no two sides parallel. Thus, *A* is a trapezium.

3. A **trapezoid** is a quadrilateral having at least two sides parallel. Thus, *B*, *C*, *D*, *E*, *F*, are trapezoids.

4. A **parallelogram** is a trapezoid having its opposite sides parallel. Thus, *C*, *D*, *E*, *F*, are parallelograms.

5. A **rhomboid** is an oblique parallelogram. Thus, *C* and *D* are rhomboids.

6. A **rhombus** is an équilateral rhomboid. Thus, *D* is a rhombus.

7. A **rectangle** is a right parallelogram. Thus, *E* and *F* are rectangles.

8. A **square** is an equilateral rectangle. Thus, *F* is a square.

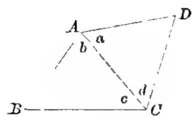
9. Two parallel sides of a trapezoid are called *bases.*

10. The base on which it is supposed to stand is called its *lower base*; the base opposite, the *upper base.*

11. The **altitude** of a trapezoid is the perpendicular distance between its bases.

78. Proposition XXXII.—Theorem.

The sum of the four angles of any quadrilateral is equal to four right angles.



Let R denote a right angle, and draw AC, dividing $ABCD$ into two triangles. Then,

$$a + D + d = 2R,$$

$$b + B + c = 2R \text{ (52)}.$$

$$\therefore \quad a + b + B + c + d + D = 4R.$$

But, $a + b = A$, $c + d = C$; $\therefore$ $A + B + C + D = 4R$.

79. Corollary.

If two angles of a quadrilateral are supplemental, the other two angles are supplemental. (?)

80. Proposition XXXIII.—Theorem.

The opposite angles of any parallelogram are equal.

For, the sides of the angles, A, C, are parallel and lie in opposite directions; hence $A = C$ (48, 2). In like manner, it is proved that $B = D$.

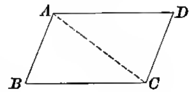
81. Corollary.

If one angle of a parallelogram is a right angle, each of its other angles is a right angle. (?)

82. Proposition XXXIV.—Theorem.

The opposite sides of a parallelogram are equal.

For, drawing the diagonal AC, the alternate angles, ACB, CAD, are equal (**46,** 2); also, BAC, ACD. Then, the triangles, ACB, ACD, are equal (**57**); $\therefore$ $AB = CD$, and $AD = BC$ (**56,** 2).



83. Corollaries.

1. *A diagonal of a parallelogram divides it into two equal triangles.*

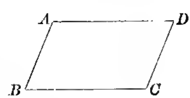
2. *Two parallelograms are equal if two sides and the included angle of the one are respectively equal to two sides and the included angle of the other* (**21,** 5).

3. *Two rectangles are equal if they have equal bases and equal altitudes* (**83,** 2).

84. Proposition XXXV.—Theorem.

A quadrilateral is a parallelogram if its opposite angles are equal.

Let $ABCD$ be a quadrilateral having its opposite angles equal. By hypothesis, we have.



$$A = C, \quad B = D; \quad \therefore \quad A + B = C + D.$$

But, $\quad A + B + C + D = 4R$ (**78**).

$\therefore \quad 2(A + B) = 4R, \quad \therefore \quad A + B = 2R.$

Hence the sides, AD, BC, are parallel (**43**).

$$A = C, \quad D = B; \quad \therefore \quad A + D = C + B.$$

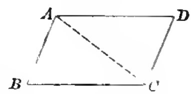
But, $\quad A + D + C + B = 4R.$

$\therefore \quad 2(A + D) = 4R, \quad \therefore \quad A + D = 2R.$

Hence the sides, AB, CD, are parallel (**43**). Therefore $ABCD$ is a parallelogram (**77**, 4).

85. Proposition XXXVI.—Theorem.

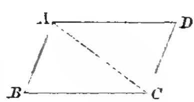
A quadrilateral is a parallelogram if its opposite sides are equal.



Let $ABCD$ be a quadrilateral having its opposite sides equal. Drawing the diagonal AC, the triangles, ABC, ADC, having AC common, $AB = CD$ and $AD = BC$ by hypothesis, are equal (**75**). Hence, the angle ACB is equal to the angle CAD (**56**, 2). $\therefore$ AD and BC are parallel (**44**, 2). In like manner, it is proved that AB and CD are parallel. Hence $ABCD$ is a parallelogram (**77**, 4).

86. Proposition XXXVII.—Theorem.

A quadrilateral is a parallelogram if two of its opposite sides are equal and parallel.



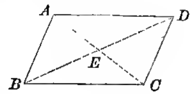
Let $ABCD$ be a quadrilateral having AD, BC, equal and parallel. Drawing AC, the angles, ACB, CAD, are equal (**46**, 2). Then, the triangles, ACB, ACD, are equal (**55**). Hence, the angles, BAC, ACD, are equal. (?) Therefore, AB, CD, are parallel (**44**, 2). Hence $ABCD$ is a parallelogram (**77**, 4).

87. Corollary.

Two lines are parallel if two points of either are equally distant from the other. (?)

88. Proposition XXXVIII.—Theorem.

The diagonals of a parallelogram bisect each other.



Let $ABCD$ be a parallelogram, AC, BD, its diagonals intersecting at E. $AD = BC$ (**82**), the angles, ADE, CBE, are equal (**46,** 2), also, the angles, DAE, BCE. Hence, the triangles, AED, BEC, are equal (**57**). Therefore, the sides, AE, CE, are equal (**56,** 2), also, the sides, BE, DE. Hence, the diagonals bisect each other.

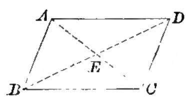
89. Corollaries.

1. *The diagonals of a rhombus or a square bisect each other at right angles* (**70,** 3).

2. *Each diagonal of a rhombus or a square bisects two opposite angles* (**60,** 3).

90. Proposition XXXIX.—Theorem.

A quadrilateral is a parallelogram if its diagonals bisect each other.



Let $ABCD$ be a quadrilateral whose diagonals bisect each other. Then, $ABCD$ is a parallelogram.

For, $AE = CE$, $DE = BE$, by hypothesis, and the angles, AED, BEC, are equal (**33**). Hence, the triangles, AED, BEC, are equal (**55**), and the angles, ADB, CBD, are equal (**56,** 2). Therefore, the sides, AD, CB, are parallel (**44,** 2).

In like manner, it is proved that AB, CD, are parallel. Hence, $ABCD$ is a parallelogram (**77,** 4).

91. Corollary.

A quadrilateral is a rhombus or a square if its diagonals bisect each other at right angles. (?)

92. Exercises.

1. If two adjacent angles of a quadrilateral are supplemental, the quadrilateral is a trapezoid.

2. If the angles adjacent to one base of a trapezoid are equal, the angles adjacent to the other base are equal.

3. The diagonals of a rectangle are equal.

4. A parallelogram is a rectangle if its diagonals are equal.

5. If on passing round the perimeter of a square four points be taken equally distant from the vertices left, and these points be joined in order, two and two, the figure formed will be a square.

6. If from a variable point in the base of an isosceles triangle parallels to the equal sides be drawn, a parallelogram is formed, whose perimeter is equal to the sum of the equal sides of the triangle.

V. POLYGONS IN GENERAL.

93. Definitions and Classification.

1. A **plane polygon** is a plane figure bounded by straight lines.

2. Polygons are classified in reference to the number of angles, as *trigons* (*triangles*), *tetragons* (*quadrilaterals*), *pentagons*, *hexagons*, *heptagons*, *octagons*, *enneagons*, *decagons*, *hen-*

decagons, dodecagons, decatrigons, decatetragons, decapentagons, decahexagons, decaoctagons, decaenneagons, icosagons . . . .

The number of sides of a polygon is equal to the number of angles.

3. A **pentagon** is a polygon of five sides.

4. In like manner, define a *hexagon*, a *heptagon*, etc., up to an *icosagon.*

5. An **equilateral polygon** is a polygon which has all its sides equal.

6. An **equiangular polygon** is a polygon which has all its angles equal.

7. A **regular polygon** is a polygon which is both equilateral and equiangular.

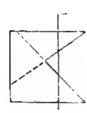
8. **Mutually equilateral polygons** are polygons which have their corresponding sides equal.

9. **Mutually equiangular polygons** are polygons which have their corresponding angles equal.

10. **Equal polygons** are those which can be made to coincide.

11. A **convex polygon** is a polygon of which each interior angle is less than two right angles.

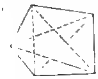
It is evident that no side of a convex polygon will, when produced, divide the polygon, and that a straight line can not intersect the perimeter in more than two points.



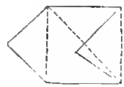
12. A **concave polygon** is a polygon of which at least one interior angle is greater than two right angles.

Such an angle is called *re-entrant,* and each of its sides produced through the vertex will divide

the polygon. A straight line may intersect the perimeter of a concave polygon in more than two points.



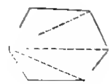
13. The diagonals of a convex polygon are all interior.



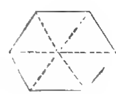
14. A diagonal of a concave polygon may be interior, exterior, or partly interior and partly exterior.

15. A polygon is to be regarded convex unless it is stated to be concave.

16. A polygon can be divided into triangles,



1st. By drawing diagonals from the vertex of one of its angles to the vertices of all the angles not adjacent. In this case, the number of triangles is equal to the number of sides of the polygon minus two.



2d. By drawing lines from a point within to the vertices of the angles. In this case, the number of triangles is equal to the number of sides.

94. Proposition XL.—Theorem.

Equal polygons can be divided by diagonals into the same number of triangles respectively equal and similarly arranged.

For, since these polygons are equal, one may be placed on the other so that they shall coincide.

Then, diagonals drawn from the common vertex of any two coincident angles to the common vertices of all the angles not adjacent will divide the coincident polygons into coincident triangles, which are therefore equal and similarly arranged.

95. Proposition XLI.—Theorem.

Polygons are equal if they can be divided by diagonals into the same number of triangles respectively equal and similarly arranged.

For, placing one on the other, the equal parts being similarly arranged can be made to coincide. Hence, the polygons will coincide, and are therefore equal.

96. Proposition XLII.—Theorem.

Equal polygons are mutually equilateral and mutually equiangular.

For, since they are equal, they can be made to coincide. Hence, the coincident sides and angles are equal; that is, the polygons are mutually equilateral and mutually equiangular.

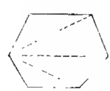
97. Proposition XLIII.—Theorem.

Polygons are equal if they are mutually equilateral and mutually equiangular, the equal parts being similarly arranged.

For, placing one on the other, the equal parts can be made to coincide. Hence, the polygons can be made to coincide, and are therefore equal.

98. Proposition XLIV.—Theorem.

The sum of all the angles of a polygon is equal to two right angles taken as many times less two as the polygon has sides.



For, let n denote the number of sides, R one right angle, and s the sum of the angles. Then, by drawing diagonals from the vertex of one of the angles to the vertices of all the angles not adjacent, it is evident that the

polygon will be divided into $n - 2$ triangles. It is also evident that the sum of all the angles of all the triangles is equal to the sum of all the angles of the polygon.

But the sum of all the angles of one triangle is $2R$. Hence, the sum of all the angles of the $n - 2$ triangles, which is the sum of all the angles of the polygon, is equal to $n - 2$ times $2R$, which is $2R(n - 2)$. Hence,

$$s = 2R(n - 2).$$

99. Corollaries.

1. If the polygon is equiangular, denote each angle by A; then, since there are as many angles as sides, $n =$ the number of angles, and since each angle is $\frac{1}{n}$ of the sum,

$$A = \frac{2R(n-2)}{n}.$$

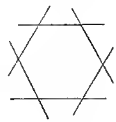
2. Since, in a triangle, $n = 3$, in a quadrilateral, $n = 4$, etc., we have s for all cases, and A for equiangular polygons:

In general,	$s = 2R(n - 2)$,	$A = \frac{2R(n-2)}{n}$.
In a triangle,	$s = 2R$,	$A = \frac{2}{3}R$.
In a quadrilateral,	$s = 4R$,	$A = R$.
In a pentagon,	$s = 6R$,	$A = \frac{6}{5}R$.
In a hexagon,	$s = 8R$,	$A = \frac{4}{3}R$.
. . . . . .	. . .	. . . .

100. Proposition XLV.—Theorem.

If each side of any polygon be produced in both directions, the sum of all the angles, interior and exterior, will be equal to four right angles taken as many times as the polygon has sides.

Let R denote one right angle, n, the number of sides or angles, I, the sum of the interior angles, E, the sum of the exterior angles, V, the sum of the exterior angles vertical to the interior, A, the sum of the exterior angles adjacent to the interior. Then, since the sum of the angles about one vertex is $4R$, the sum of the angles about the n vertices will be n times $4R$, which is $4nR$. But $I + E$ is equal to this sum. Hence, $I + E = 4nR$.



101. Corollaries.

1. *The sum of the exterior angles is* $2nR + 4R$.

For, $E = 4nR - I = 4nR - 2R(n - 2) = 2nR + 4R$.

2. *The sum of the exterior angles vertical to the interior is* $2R(n - 2)$.

For, $V = I = 2R(n - 2)$.

3. *The sum of the exterior angles adjacent to the interior is* $8R$.

For, $A = E - V = 2nR + 4R - 2R(n - 2) = 8R$.

4. *If but one exterior angle adjacent to the interior be taken at each vertex, their sum will be* $4R$. (?)

102. Exercises.

1. Prove that the sum of the interior angles of a polygon of n sides is equal to $2nR - 4R$, by drawing lines from a point within to the vertices of the polygon.

2. If each side of a polygon be produced in one direction forming one exterior angle at each vertex, prove that the sum of these exterior angles is $4R$, by drawing, from any point in the plane of the polygon, lines respectively parallel to the sides.

3. What is the greatest number of interior acute angles that any convex polygon can have?

4. Prove that the whole number of diagonals that can be drawn in a polygon of n sides is $\frac{1}{2}n(n-3)$.

5. Find from the above formula the number of diagonals that can be drawn in a quadrilateral, in a pentagon, in a hexagon, and so on, to an icosagon; also, in a triangle.

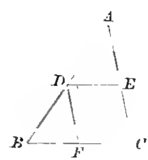
6. What regular polygons of the same kind can be used in making a pavement? (**34,** 2), (**99,** 2).

7. What combinations of different regular polygons can be used in making a pavement?

VI. SUPPLEMENTARY PROPOSITIONS.

103. Proposition XLVI.—Theorem.

The straight line drawn from the middle point of one side of a triangle, parallel to a second side and terminating in the third, bisects the third side, and is equal to one-half the second.



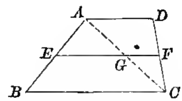
In the triangle ABC, let DE be drawn from D, the middle point of AB, parallel to BC and terminating in AC. Draw DF parallel to AC; then, in the triangles, ADE, DBF, $AD = DB$, $ADE = DBF$, $DAE = BDF$ (**46,** 4); hence, the triangles, ADE, DBF, are equal (**57**); $\therefore$ $AE = DF$. But, $DF = EC$, since $DECF$ is a parallelogram (**82**); $\therefore$ $AE = EC$; hence, DE bisects AC.

In the equal triangles, ADE, DBF, $DE = BF$; (?) but $DE = FC$; (?) $\therefore$ $BF = FC$; $\therefore$ $BF = \frac{1}{2}BC$, $\therefore$ $DE = \frac{1}{2}BC$.

104. Corollaries.

1. *The straight line joining the middle points of two sides of a triangle is parallel to the third side and equal to its half.* (?)

2. *The straight line drawn from the middle point of one of the non-parallel sides of a trapezoid, parallel to the bases, and terminating in the other non-parallel side, bisects that side, and is equal to one-half the sum of the bases.*



In the trapezoid $ABCD$, let EF be drawn from E, the middle point of AB, one of the non-parallel sides, parallel to the bases, and terminating in the other non-parallel side. Draw AC intersecting EF in G; then, since E is the middle point of AB, and EG is parallel to BC, G is the middle point of AC, and $EG = \frac{1}{2}BC$ (**103**).

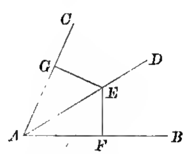
Since G is the middle point of AC, and GF is parallel to AD; F is the middle point of DC, and $GF = \frac{1}{2}AD$. (?)

$\therefore \; EG + GF = \frac{1}{2}(BC + AD), \; \therefore \; EF = \frac{1}{2}(BC + AD).$

3. *The straight line joining the middle points of the non-parallel sides of a trapezoid is parallel to the bases, and equal to one-half their sum.* (?)

105. Proposition XLVII.—Theorem.

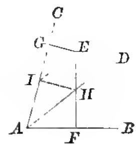
Every point in the bisector of an angle is equally distant from the sides of the angle.



Let E be any point of AD, the bisector of the angle BAC, and EF, EG, the perpendiculars from E to AB, AC, respectively. The right triangles, AEF, AEG, are equal (**72,** 2), and $EF = EG$ (**56,** 2). Hence, E is equally distant from the sides of the angle BAC (**71**).

106. Proposition XLVIII.—Theorem.

Every point within an angle, not in the bisector, is unequally distant from the sides of the angle.



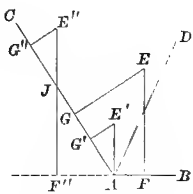
1. Let BAC be right or acute, AD the bisector, E a point within the angle, but not in the bisector, EF, EG, perpendiculars to AB, AC, respectively. One of these perpendiculars will intersect the bisector in H. Let fall HI perpendicular to AC, and draw IE.

Since EG is perpendicular to AC, EI is oblique (**30**);

$\therefore \quad EG < EI$ (**69**, 1); but $EI < EH + HI$ (**65**);

$\therefore \quad EG < EH + HI$; but $HI = HF$ (**105**);

$\therefore \quad EG < EH + HF, \quad \therefore \quad EG < EF.$



2. Let the angle BAC be obtuse, AD the bisector, E, E', E'', points within the angle, not in the bisector.

If the perpendiculars, EF, EG, fall on the sides of the angle, respectively, the proof is the same as when the angle is acute.

If one perpendicular $E'A$ fall at the vertex of the angle, then, since $E'G'$ is perpendicular to AC, $E'A$ is oblique; $\therefore \; E'G' < E'A$.

If one perpendicular $E''F''$ fall on one side produced through the vertex, it will cut the other side in some point J. Then, since $E''G''$ is perpendicular to AC, $E''J$ is oblique; $\therefore \; E''G'' < E''J$; but $E''J < E''F''$;

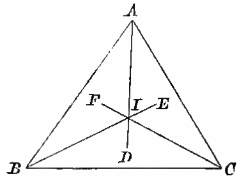
$\therefore \quad E''G'' < E''F''$.

107. Corollary.

The bisector of an angle is the locus of all the points within the angle which are equally distant from its sides. (?)

108. Proposition XLIX.—Theorem.

The bisectors of the angles of a triangle meet in the same point.



Let ABC be a triangle, AD, BE, CF, the bisectors of the angles, A, B, C, respectively. Any two bisectors will meet within the triangle. (?) Hence, I the intersection of AD and BE is within each angle.

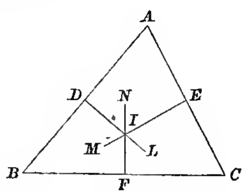
I being in the bisector AD is equally distant from AB, AC.
I being in the bisector BE is equally distant from AB, BC.
Therefore, I is equally distant from AC, BC, and hence is in the bisector CF (**107**); that is, the bisector CF passes through I, the point of intersection of the other bisectors; hence, the three bisectors meet in the same point.

109. Corollary.

The three perpendiculars from the common intersection of the bisectors of the angles of a triangle to the sides are equal (**105**).

110. Proposition L.—Theorem.

The perpendiculars in the plane of a triangle, to its three sides, at their middle points, meet in the same point.



Let ABC be a triangle, DL, EM, FN, the perpendiculars to the three sides in their plane, at their middle points.

Any two of these perpendiculars, as DL, EM, will meet if sufficiently produced; for, if they do not meet, they are parallel (**40**). Then, AB, which by

hypothesis is perpendicular to DL, is also perpendicular to EM parallel to DL (**42**, 3). But, by hypothesis, AC is perpendicular to EM. Now, AB and AC are different lines, since they are sides of the triangle. Then we have two perpendiculars, AB, AC, let fall from the same point A to the same straight line EM, which is impossible (**30**). Hence, DL and EM are not parallel, and must therefore meet in the same point I, if sufficiently produced.

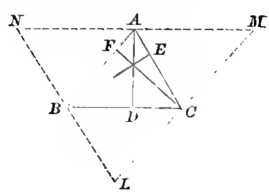
Now, I is equally distant from A and B (**69**, 2). I is also equally distant from A and C. Hence, I is equally distant from B and C, the extremities of BC, and is therefore in the perpendicular FN erected to BC at its middle point (**70**, 2). Hence, the three perpendiculars meet in the same point.

111. Corollary.

The three straight lines drawn from the intersection of the perpendiculars to the three sides of a triangle, at their middle points, to the three vertices are equal. (?)

112. Proposition LI.—Theorem.

The three perpendiculars from the vertices of a triangle to the opposite sides meet in the same point.

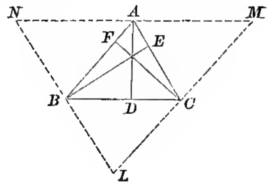


Let ABC be a triangle, AD, BE, CF, perpendiculars from the vertices, A, B, C, to the opposite sides respectively.

Through each vertex draw a line parallel to the opposite side, thus forming a new triangle LMN.

$ABCM$ and $ACBN$ are parallelograms; (?) $\therefore$ $AM = BC$, and $AN = BC$; (?) $\therefore$ $AM = AN$. In like manner, it can

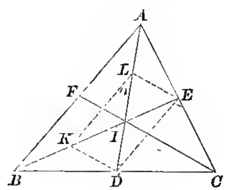
be proved that $BN = BL$, and $CM = CL$. Hence, the points, A, B, C, are respectively the middle points of the sides MN, NL, LM, of the triangle LMN.



Now AD, which is perpendicular to BC, is perpendicular to MN, which is parallel to BC (**42**, 3). In like manner, it can be proved that BE is perpendicular to NL, and that CF is perpendicular to LM. Hence, AD, BE, CF, the three perpendiculars from the vertices of the triangle ABC to the opposite sides, are also the perpendiculars to the sides of the triangle LMN at their middle points, and therefore meet in the same point (**110**).

113. Proposition LII.—Theorem.

The three medial lines of a triangle meet in the same point.



Let ABC be a triangle, AD, BE, CF, the three medial lines (**51**, 17). Since D lies between B and C, AD lies between AB and AC. Also, BE lies between BA and BC, and therefore intersects AD at some point I within the triangle. Join L and K, the middle points of AI and BI respectively, and draw LE, ED, DK.

In the triangle AIB, LK joining the middle points of AI and BI, is parallel to AB and equal to $\frac{1}{2}AB$ (**103**).

In the triangle ACB, ED joining the middle points of AC and BC, is parallel to AB and equal to $\frac{1}{2}AB$.

Hence, LK and ED are parallel and equal; (?) therefore, $LKDE$ is a parallelogram (**86**); hence, its diagonals bisect

each other (88); $\therefore$ $DI = IL = LA$, $\therefore$ $DI = \frac{1}{3}DA$. Hence, the medial line BE intersects the medial line AD in a point whose distance from D is $\frac{1}{3}DA$.

In like manner, it is proved that CF intersects AD in the same point. Therefore, the three medial lines intersect in the same point.

114. Exercises.

1. The straight line joining the middle points of the adjacent sides of any quadrilateral form a parallelogram whose perimeter is equal to the sum of the diagonals of the quadrilateral (**104**, 1).

2. The lines joining the middle points of the opposite sides of any quadrilateral bisect each other.

3. The three straight lines joining the middle points of the sides of any triangle divide the triangle into four equal triangles.

4. The straight line joining the middle points of the lines joining the alternate extremities of two parallel lines is equal to one-half the difference of the parallels.

5. If any two sides of a triangle be produced across the third side, the bisectors of the exterior angles thus formed will intersect on the bisector of the angle included by the sides produced (**107**).

6. The medial line to any side of a triangle is less than the half sum of the other sides and greater than the difference of that half sum and half the third side (**88**), (**82**), (**65**), (**66**).

7. The sum of the three medial lines of a triangle is less than the perimeter and greater than half the perimeter of the triangle.

8. The two lines drawn from the intersection of the bisectors of two angles of any equilateral triangle to the common side of the bisected angles, respectively parallel to the other sides, trisect the common side.

9. The two straight lines drawn from one vertex of a parallelogram to the middle points of the opposite sides trisect a diagonal.

10. The four bisectors of the angles of any quadrilateral form a second quadrilateral whose opposite angles are supplemental. If the first quadrilateral is a parallelogram, the second is a rectangle; if the first is a rectangle, the second is a square; if the first is a square, the second vanishes.

11. The common intersection of the three perpendiculars, in the plane of a triangle, to its sides at their middle points, is within the triangle if acute, on the hypotenuse if right, without the triangle if obtuse.

12. The sum of the three perpendiculars from any point within an equilateral triangle to the sides, is equal to the altitude of the triangle.

BOOK II.

I. STRAIGHT LINES AND CIRCLES.

115. Definitions.

1. A **circle** is a plane figure bounded by a line all the points of which are equally distant from a point within.

2. The **circumference** of a circle is the line which bounds the circle.

3. The **center** of a circle is the point within from which all the points of the circumference are equally distant.

4. A **radius** of a circle is a straight line having the center for its origin and any point of the circumference for its extremity.

5. A **diameter** of a circle is any straight line passing through the center and having its extremities in the circumference.

6. An **arc** of a circle is any portion of the circumference.

7. A **semi-circumference** is an arc equal to one-half the circumference.

8. A **chord** of a circle is any straight line having its extremities in the circumference. Every chord subtends two arcs whose sum is the circumference. If the arcs subtended by a chord are unequal, the less arc is to be understood as the arc of the chord, unless it is otherwise stated.

9. A **segment** of a circle is a portion of the circle inclosed by an arc and its chord.

10. A **semicircle** is a segment equal to one-half the circle.

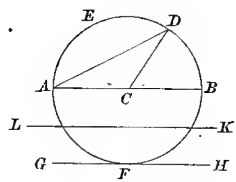
11. A **sector** of a circle is a portion of the circle inclosed by an arc and the two radii drawn to the extremities of the arc.

12. A **tangent** is an indefinite straight line in the plane of the circle which has only one point in common with the circumference.

13. The **point of tangency** is the point common to the circumference and tangent.

14. A **secant** is a straight line which intersects the circumference in two points, and lies partly within and partly without the circle.

15. In the annexed figure, the portion of the plane bounded by $AEBF$ is a circle, the boundary is the circumference, C is the center, CD is a radius, AB is a diameter, AED is an arc, AD is a chord, ADE is a segment, DCB is a sector, GH is a tangent, F the point of tangency, LK is a secant, AFB is a semi-circumference, and ABF is a semicircle. This circle may be read, the circle C.



116. Proposition I.—Theorem.

The surface generated by a line revolving in the same plane about one of its extremities fixed as a center, is a circle.

For, the revolving line generates a plane surface of which the revolving extremity generates the boundary, every point

of which is at a distance from the fixed extremity equal to the length of the revolving line; hence, the surface generated is a circle (**115,** 1), the line generated by the revolving extremity is the circumference, the fixed extremity is the center, the revolving line is the radius.

117. Corollaries.

1. *All the radii of a circle are equal.* (?)

2. *All the diameters of a circle are equal.* (?)

3. *The circumference of a circle is a curved line.*

For, all of its points are at a distance from the center equal to the radius; but a straight line can not have more than two points equally distant from a given point (**70,** 1); hence, no three points of the circumference lie in a straight line. Therefore, the circumference continually changes its direction, and hence is a curved line (**3,** 6).

4. *Circles having equal radii are equal.* (?)

5. *The locus of all the points in a plane, at a given distance from a given point in the plane, is the circumference of the circle whose center is the given point, whose radius is the given distance, and whose plane is the given plane.* (?)

118. Proposition II.—Theorem.

A straight line can not intersect the circumference of a circle in more than two points.

For, if possible, let a straight line intersect a circumference in three points. The three radii drawn to these points are equal (**117,** 1). Then we should have three equal straight lines drawn from the same point to the same straight line,

which is impossible (**70**, 1). Hence, the supposition that a straight line can intersect the circumference of a circle in three points, which led to this impossibility, is false (**15**, 4). Therefore, a straight line can not intersect a circumference in more than two points.

119. Proposition III.—Theorem.

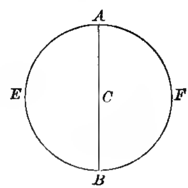
An indefinite straight line which intersects the circumference of a circle in one point, intersects it in another point.

For, since the straight line intersects the circumference, it passes within, and, if produced sufficiently, must pass without, and hence intersect the circumference in another point.

120. Proposition IV.—Theorem.

Any diameter bisects the circle and its circumference.

Let C be a circle, AB its diameter. Then, AB bisects the circle and its circumference.

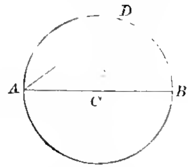


For, let AEB revolve about AB as an axis, till the planes of the two segments of the circle coincide. Then, since all the points of the two arcs, AEB, AFB, are equally distant from the center, these arcs will coincide, and hence the segments will coincide; therefore, the arcs are equal and the segments are equal.

121. Proposition V.—Theorem.

A diameter is greater than any chord which does not pass through the center.

Let C be a circle, AB its diameter, AD a chord not passing through the center. Then, $AB > AD$.



For, the chord AD, not passing through the center, and the radii, CA, CD, drawn to the extremities of the chord, form a triangle ACD.

Now, $\quad AC + CD > AD$ (65); but $CD = CB$ (**117**, 1);

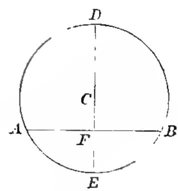
$\therefore \quad AC + CB > AD$; but $AC + CB = AB$;

$$\therefore \quad AB > AD.$$

122. Proposition VI.—Theorem.

The diameter perpendicular to a chord bisects the chord and the arcs subtended by the chord.

Let C be a circle, and DE a diameter perpendicular to the chord AB. Then, DE bisects AB and the subtended arcs, AEB, ADB.



For, revolve the semi-circumference DAE about DE as an axis, till it coincides with the semi-circumference DBE.

Since DE is perpendicular to AB, FA will fall on FB, and A, the point common to DAE and FA, will be found in DBE and FB, and hence at their intersection B. Then, FA coincides with FB, EA with EB, DA with DB.

$$\therefore \quad FA = FB, \quad EA = EB, \quad DA = DB.$$

123. Corollaries.

1. *The perpendicular to a chord at its middle point passes through the center of the circle and bisects the arcs subtended by the chord.* (?)

2. *The intersection of the perpendiculars to two chords at their middle points is the center of the circle.* (?)

3. *The straight line which passes through any two of the four points, the center of the circle, the middle point of a chord, the middle points of the arcs subtended by the chord, passes through the other points, and is coincident with the diameter perpendicular to the chord.* (?)

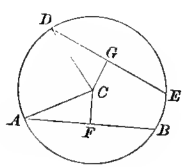
4, *The locus of the middle points of a system of parallel chords is the diameter perpendicular to these chords, and the extremities of this diameter are the middle points of all arcs subtended by these chords.* (?)

5. *A diameter perpendicular to a chord bisects the angle formed by two lines drawn from any point of the diameter to the extremities of the chord.* (?)

6. *The distance of a chord from the center is the straight line having the center and the middle point of the chord for its extremities.* (?)

124. Proposition VII.—Theorem.

In the same circle or in equal circles, equal chords are at equal distances from the center.



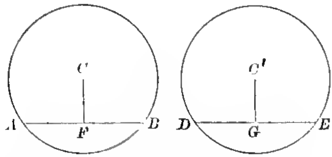
Let C be the center of a circle, AB, DE, equal chords. Then, the perpendiculars, CF, CG, which measure the respective distances of the chords from the center, are equal.

For, these perpendiculars bisect the chords (**122**); $\therefore$ $AF = DG$. (?) Hence, the right triangles, AFC, DGC, are equal (**72**, 1); $\therefore$ $CF = CG$.

In case of equal circles, the demonstration is similar.

125. Proposition VIII.—Theorem.

In the same circle or in equal circles, chords at equal distances from the center are equal.



Let C and C' be the centers of equal circles, AB, DE, chords at equal distances from the centers. Then, $AB = DE$.

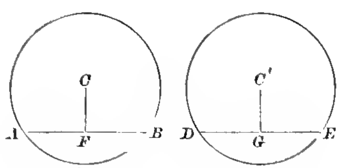
For, CF, $C'G$, perpendicular respectively to AB, DE, are equal by hypothesis.

Now, let the circle C be placed on the circle C', so that the centers coincide; also, CF and $C'G$. Then, since AB is perpendicular to CF, and DE to $C'G$, AB will fall on DE, A on D, B on E; otherwise, these points would be at unequal distances from the center. Hence, AB and DE coincide, and are therefore equal.

In case the chords are at equal distances from the center of the same circle, each is equal to a chord at an equal distance from the center of an equal circle, and hence they are equal to each other.

126. Proposition IX.—Theorem.

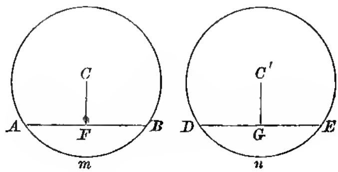
In the same circle or in equal circles, equal arcs are subtended by equal chords; and conversely, equal chords subtend equal arcs.



Let C, C', be the centers of equal circles, AmB, DnE, equal arcs. Then, $AB = DE$.

For, let the circle C be placed on the circle C', so that the equal arcs, AmB, DnE, coincide. Then,

A falls on D, and B on E; hence, AB and DE coincide, and are therefore equal.



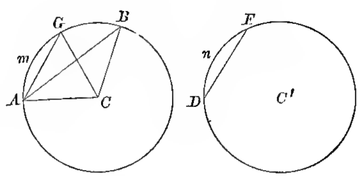
Conversely, let AB be equal to DE; then, $AmB = DnE$.

For, the perpendiculars, CF, $C'G$, from the centers to the chords, bisect the chords (**122**), and are equal (**124**).

Now, let the circle C be made to coincide with the circle C', and CF with $C'G$; then, the equal chords, AB, DE, will coincide; hence, AmB coincides with DnE; (?) therefore, $AmB = DnE$.

127. Proposition X.—Theorem.

In the same circle or in equal circles, the greater of two unequal arcs is subtended by the greater chord; and conversely, the greater of two unequal chords subtends the greater arc.



Let C and C' be equal circles, and

$$AmB > DnE.$$

Then, $AB > DE$.

Take $AmG = DnE$, and draw AG, CA, CB, CG. Since $AmG = DnE$, $AG = DE$ (**126**). The angle $ACB > ACG$ (**21**, 6). Then, in the triangles, ACB, ACG, $AB > AG$ (**73**); $\therefore$ $AB > DE$.

Conversely, let $AB > DE$, then $AmB > DnE$; for, if $AmB = DnE$, $AB = DE$; (?) if $AmB < DnE$, $AB < DE$. (?) Both of these results are contrary to the hypothesis that $AB > DE$; $\therefore$ neither $AmB = DnE$ nor $AmB < DnE$; $\therefore$ $AmB > DnE$.

128. Scholiums.

1. Each of the arcs considered in the last proposition is supposed to be less than a semi-circumference.

2. If each arc is greater than a semi-circumference, the greater arc is subtended by the less chord, and conversely.

3. If one arc is greater and the other less than a semi-circumference, then,

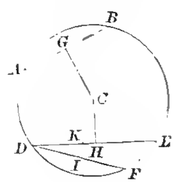
1st. If the sum of the arcs is less than a circumference, the greater arc is subtended by the greater chord, and conversely.

2d. If the sum of the arcs is equal to a circumference, the arcs are subtended by equal chords, and conversely.

3d. If the sum of the arcs is greater than a circumference, the greater arc is subtended by the less chord, and conversely.

129. Proposition XI.—Theorem.

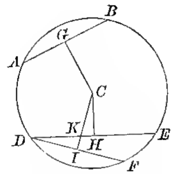
In the same circle or in equal circles, if two chords are unequal, the less is at the greater distance from the center; and conversely, if two chords are at unequal distances from the center, that which is at the greater distance is the less.



Let C be the center of a circle, the chord AB less than the chord DE; then, CG, perpendicular to AB, is greater than CH, perpendicular to DE. If DE is a diameter, it passes through the center; $\therefore$ $CH = 0$; and since $AB < DE$, AB is not a diameter;

$$\therefore \; CG > 0; \; \therefore \; CG > CH.$$

If DE is not a diameter, on the less of the two arcs subtended by DE take the arc DF equal to the arc AB, draw the chord DF, and CI perpendicular to DF. $DF = AB$ (**126**), and $CI = CG$ (**124**).



The chord DE intersects CI in some point K. (?)

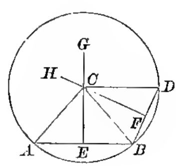
Since DI is perpendicular to CI, DK is oblique to CI (**30**); that is, CK is oblique to DE; but CH is perpendicular to DE; $\therefore$ $CK > CH$ (**69**, 1); but $CI > CK$; $\therefore$ $CI > CH$; but $CG = CI$; $\therefore$ $CG > CH$.

Conversely, let $CG > CH$; then, $AB < DE$; for, if $AB = DE$, $CG = CH$ (**124**); if $AB > DE$, $CG < CH$; (?) but both these results are contrary to the hypothesis that $CG > CH$; $\therefore$ neither $AB = DE$ nor $AB > DE$;

$$\therefore\ AB < DE.$$

130. Proposition XII.—Theorem.

Through any three points not in the same straight line a circumference of a circle can be made to pass.



Let A, B, D, be three points not in a straight line. Draw the straight lines, AB, BD, and in the plane of the points erect the perpendiculars, EG, FH, to AB, BD, respectively, at their middle points.

These perpendiculars will meet; for, if not, they are parallel (**40**); then, AB, perpendicular to EG, would be perpendicular to FH, parallel to EG (**42**, 3); but BD is perpendicular to FH. (?)

Then, since the three points, A, B, D, are not in a straight line, we have two straight lines, AB, BD, drawn from the same point B, perpendicular to the same straight line FH, which is impossible (**30**). Hence, EG and FH must meet in some point C.

Now, C being in EG, perpendicular to AB, at its middle point is equally distant from A and B; that is, $CA = CB$ (**69**, 2).

Also, C being in FH, perpendicular to BD, at its middle point is equally distant from B and D; that is, $CB = CD$.

Hence, C is equally distant from the three points, A, B, D.

If, therefore, a circumference be described having C for a center and CA for a radius, it will pass through the points, A, B, D.

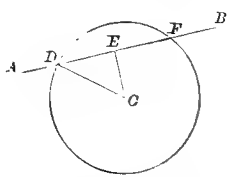
131. Corollaries.

1. *All the circumferences which pass through three points not in the same straight line are coincident.* (?)

2. *Two circumferences can intersect in only two points.* (?)

132. Proposition XIII.—Theorem.

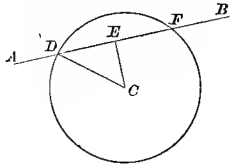
An indefinite straight line oblique to a radius at its extremity is a secant; and conversely, a secant is oblique to the radius drawn to either point of intersection.



Let AB be oblique to the radius CD, at its extremity D; then, AB is a secant.

For, let CE be perpendicular to AB (**30**); then, $CE < CD$ (**69**, 1). But CD is a radius; hence, CE is less than a radius; therefore, the point E is within the

circumference. Now, AB having the point D in common with the circumference (**115,** 4), and the point E within, intersects the circumference in one point D, and hence, in another point F (**119**), and lies partly within and partly without the circle, and is therefore a secant (**115,** 14).

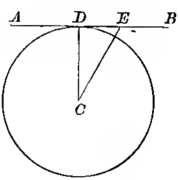


Conversely, let AB be a secant, and CD a radius drawn to D, one point of intersection of the secant and circumference. Since the secant passes within the circumference, it has points nearer the center than D; hence, CD is not the perpendicular from C to AB; therefore, AB is oblique to CD.

133. Proposition XIV.—Theorem.

A straight line perpendicular to a radius at its extremity is a tangent to the circumference; and conversely, a tangent to a circumference is perpendicular to the radius drawn to the point of tangency.

Let AB be perpendicular to the radius CD, at its extremity D. Then, AB is a tangent.



For, since CD is perpendicular to AB, any straight line drawn from C to any other point of AB, as E, is oblique (**30**); $\therefore$ $CE > CD$ (**69,** 1). But CD is a radius; hence, CE is greater than a radius; therefore, the point E is without the circle. Hence, AB has only one point in common with the circumference, and is therefore a tangent (**115,** 12).

Conversely, the tangent AB is perpendicular to the radius

CD drawn to the point of tangency; for, if not, it is oblique to CD, and hence enters the circle and intersects the circumference in another point, and is therefore a secant (**132**), and not a tangent (**115,** 12), which is contrary to the hypothesis; hence, the tangent AB is not oblique, and is therefore perpendicular to the radius CD, drawn to the point of tangency.

134. Corollaries.

1. *The indefinite perpendicular to the tangent at the point of tangency passes through the center of the circle.* (?)

2. *Every point of the tangent except the point of tangency is without the circle.* (?)

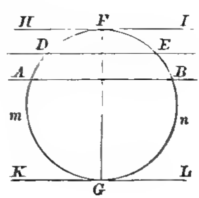
3. *At a given point of a circumference only one tangent can be drawn.* (?)

4. *If a secant revolve about either point of intersection with the circumference, the second point of intersection will move along the circumference; and when it coincides with the other, the secant becomes a tangent which may, therefore, be regarded as a secant whose points of intersection are coincident.*

135. Proposition XV.—Theorem.

Two parallels meeting a circumference intercept equal arcs.

1. When the parallels are secants, or chords:



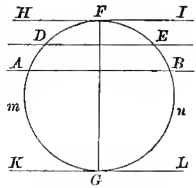
Let AB and DE be parallel secants or chords, and FG the diameter perpendicular to them. Then,

$$AF = BF, \text{ and } DF = EF \text{ (122)}.$$

$$\therefore\ AF - DF = BF - EF \text{ (23, 7)}.$$

Hence, $\quad AD = BE.$

2. When one of the parallels is a secant or a chord and the other a tangent, as AB and HI:



Let DE move toward HI, but in such a manner as to continue parallel to AB, till it coincides with HI (**42**, 5). The points D and E will approach and finally meet in F, AD will become AF, and BE, BF. But in all positions of the moving parallel, $AD = BE$. Hence, when DE becomes the tangent HI, $AF = BF$.

3. When the parallels are both tangents, as HI and KL:

Let DE move toward HI, and AB toward KL, but in such a manner as to continue parallel, till DE coincides with HI, and AB with KL. The points D and E will finally meet in F, and A and B in G, AD will become GmF, and BE, GnF. But in all positions of the moving parallels, $AD = BE$. Hence, when the parallels become tangents, $GmF = GnF$.

136. Exercises.

1. What is the locus of all the points in a given plane which are at a given distance from a given point in that plane? Does this locus embrace all the points at the given distance from the given point?

2. What is the locus of all the points which are at a given distance from the circumference in the plane of a circle? Of how many lines does this locus consist? When would one of these lines reduce to a point?

3. The least chord that can be drawn through a given point in a circle is perpendicular to the diameter through the point (**129**).

4. The angle contained by two tangents at the extremities of a chord is twice the angle contained by the chord and the diameter drawn from either extremity of the chord (**133**).

II. RELATIVE POSITION OF CIRCLES.

137. Remarks.

If two circumferences lie in the same plane, then,

1. They are *wholly external* to each other when they are separated by intervening space.

2. They are *tangent externally* when they are external to each other except one common point of tangency.

3. They *intersect* when they cross each other.

4. They are *tangent internally* when one is within the other except one common point of tangency.

5. One is *wholly within* the other when the former is inclosed by the latter, no point being common.

6. They are *concentric* when they have a common center.

7. They are *coincident* when equal and concentric.

138. Proposition XVI.—Theorem.

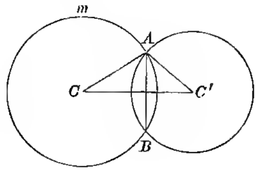
If two circumferences intersect in one point, then,

1. *They intersect in another point.*

2. *The straight line joining their centers bisects their common chord at right angles.*

3. *The distance between their centers is less than the sum of their radii and greater than their difference.*

1. Let the circumferences whose centers are C, C', intersect at A. Each has points without and within the other. Let m be a point in the circumference of C without that of C'. The circumference of C, starting from m, passing through A, and continuing in the same direction round to A again, passes within the circumference of C' at A; and since it passes through m before reaching A again, it must pass without, and hence intersect the circumference of C' in another point B.



2. Draw the common chord AB, and in the plane of the circles draw a perpendicular to AB at its middle point. This perpendicular passes through the centers, C, C' (**123,** 1), and is therefore coincident with the line joining the centers (**26,** 4); hence, the line joining the centers bisects the chord at right angles.

3. Since A and B are different points, and CC' passes through the middle point of AB, the point A is not in the line CC'; hence, ACC' is a triangle.

$$\therefore \quad CC' < CA + AC' \quad \text{and} \quad CC' > CA - AC'. \quad (?)$$

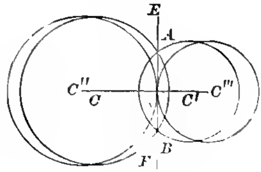
139. Proposition XVII.—Theorem.

If two circumferences are tangent to each other externally, then,

1. *The straight line joining their centers passes through their common point of tangency, and is perpendicular to their common tangent at that point.*

2. *The distance between their centers is equal to the sum of their radii.*

1. Let the circles whose centers are C, C', intersect in A and B. CC' is perpendicular to the common chord AB (**138**, 2), and hence to the common secant EF, which is the common chord produced both ways.



Now, let the circles move so that the centers may move from each other along the prolongations of CC', in such a manner that EF shall remain the common secant through the points of intersection.

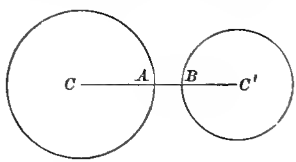
Since the chord AB is less the greater its distance from the center (**129**), and is continually bisected by the perpendicular CC' (**138**, 2), the points, A, B, approach as the circles move, and finally meet on CC', when EF becomes their common tangent perpendicular to $C''C'''$ (**115**, 12).

2. The straight line, $C''C'''$, the distance between the centers when the circumferences become tangent, passes through the common point of tangency (**139**, 1), and is, therefore, the sum of the radii.

140. Proposition XVIII.—Theorem.

If two circles are wholly external to each other, the distance between their centers is greater than the sum of their radii.

Let C and C' be the centers of two circles which are wholly external to each other. Then, CC', the distance between their centers, is greater than the sum of their radii.



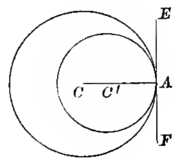
For, $CC' = CA + AB + BC'$, which exceeds the sum of the radii $CA + BC'$ by AB.

141. Proposition XIX.—Theorem.

If two circumferences are tangent to each other internally, then,

1. *The indefinite straight line passing through their centers is perpendicular to the common tangent at the point of tangency.*

2. *The distance between their centers is equal to the difference of their radii.*



1. Let the circles whose centers are C, C', be tangent to each other internally at A. EF, tangent to the greater circumference at A, has this point in common with the less circumference, since A is the common point of tangency.

Now, EF and the circumference of C' have no other point than A common; for, every point of EF, except A, is without the circumference of C (**134,** 2), and every point of the circumference of C', except A, is within the circumference of C (**137,** 4); hence, EF is tangent to the circumference of C' at A (**115,** 12).

The internal perpendicular to EF at A passes through C and C' (**134,** 1), and is therefore coincident with the line passing through the centers (**26,** 4).

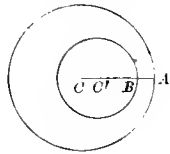
2. It is evident from the diagram that

$$CC' = CA - C'A.$$

142. Proposition XX.—Theorem.

If one circumference is wholly within another, the distance between their centers is less than the difference of their radii.

Let C and C' be the centers of two circles whose respective radii are CA and $C'B$, and let the circumference of C' be within that of C. Then,



$$CC' = CA - C'B - BA,$$

which is less than the difference of the radii by BA.

143. Exercises.

REMARK.—The first five of the following propositions can be proved by showing that the conditions are inconsistent with any other relation. Thus, in the first, the two circumferences can not be wholly external; for then the distance between their centers would be greater than the sum of their radii, which is contrary to the hypothesis. In like manner, it can be proved that no other relation, except that of intersection, is possible; hence, they intersect.

1. Two circumferences intersect, if the distance between their centers is less than the sum of their radii and greater than the difference.

2. Two circumferences are tangent externally, if the distance between their centers is equal to the sum of their radii.

3. Two circumferences are wholly external, if the distance between their centers is greater than the sum of their radii.

4. Two circumferences are tangent internally, if the distance between their centers is equal to the difference of their radii.

5. One circumference is wholly within another, if the distance between their centers is less than the difference of their radii.

6. What is the locus of the centers of all the circles having a given radius, and tangent externally to a given circumference?

7. What is the locus of the centers of all the circles having a given radius, and tangent internally to a given circumference?

8. With a given radius, describe a circumference tangent to a given circumference (**139**).

III. MEASUREMENT OF ANGLES.

144. Definitions.

1. **Ratio** is the relation of two quantities of the same kind expressed by the quotient obtained by dividing the first by the second.

2. A **measure** of a quantity is a quantity of the same kind which is contained in the given quantity a whole number of times without a remainder.

3. A **common measure** of two quantities of the same kind is a measure of each of those quantities.

4. A **unit of measure** is any measure taken as a unit.

5. A **common unit of measure** is any common measure taken as a unit.

6. To measure a quantity is to find the ratio of that quantity to the unit of measure.

7. To find the ratio of one quantity to another of the same kind is to measure the first by the second taken as the unit of measure.

8. The **numerical value** of a quantity is the ratio of that quantity to the unit of measure. If this ratio is a mixed number, the integral part is the *approximate numerical value* of the quantity.

9. The **numerical ratio** of two quantities is the ratio of their numerical values referred to a common unit.

10. **Commensurable quantities** are quantities which have a common measure.

11. **Incommensurable quantities** are quantities which have no common measure.

12. An **incommensurable ratio** is the ratio of two incommensurable quantities.

13. The **approximate numerical value** of the ratio of two incommensurable quantities is the ratio of the approximate numerical value of one to the exact or approximate numerical value of the other referred to the same unit.

14. An **inscribed angle** is an angle whose vertex is in the circumference and whose sides are chords.

15. An angle is inscribed in a segment if its vertex is in the arc of the segment and its sides pass through the extremities of the subtending chord.

16. A **central angle** is an angle whose vertex is at the center and whose sides are radii. A central angle is said to intercept the arc included between its sides.

17. **Similar arcs** are those which are intercepted by equal central angles.

18. **Similar sectors** are those which have equal central angles.

145. Proposition XXI.—Theorem.

The ratio of commensurable quantities is equal to their numerical ratio.

Let a and c be two commensurable quantities, u their common unit of measure, m the numerical value of a, and n of c, referred to their common unit of measure, u.

Then, $\frac{a}{u} = m$, and $\frac{c}{u} = n$ (**144**, 8).

Hence, $a = mu$, and $c = nu$.

$$\therefore \quad \frac{a}{c} = \frac{mu}{nu} = \frac{m}{n} \times \frac{u}{u} = \frac{m}{n} \times 1 = \frac{m}{n}.$$

146. Proposition XXII.—Problem.

To find the approximate numerical value of the ratio of two incommensurable quantities within any required degree of precision.

Let a and c be two incommensurable quantities, and let it be required to find the approximate numerical value of their ratio within $\frac{1}{n}$.

Divide c into n equal parts, and denote each part by u. Since, by hypothesis, a and c are incommensurable, u will not divide a without a remainder. Suppose that u is contained m times in a with a remainder r less than u.

Then, $\frac{a}{u} = m + \frac{r}{u}$, $\therefore\ a = mu + r$.

But, $\frac{c}{u} = n$, $\therefore\ c = nu$.

$$\therefore \quad \frac{a}{c} = \frac{mu}{nu} + \frac{r}{nu} = \frac{m}{n} + \frac{1}{n} \times \frac{r}{u}.$$

But, $\quad r < u, \therefore \frac{r}{u} < 1, \therefore \frac{1}{n} \times \frac{r}{u} < \frac{1}{n}.$

$$\therefore \quad \frac{a}{c} = \frac{m}{n} \text{ within } \frac{1}{n}.$$

As n increases, u diminishes, m increases, and $\frac{1}{n}$ diminishes. By making n sufficiently great, $\frac{1}{n}$ may be made as small as we please, and $\frac{m}{n}$ will express the approximate numerical value of the ratio of a to c within the required degree of precision.

147. Proposition XXIII.—Theorem.

Two incommensurable ratios are equal if their approximate numerical values within the same degree of precision, however exact, are always equal.

Let $\frac{a}{c}$ and $\frac{b}{d}$ be two incommensurable ratios whose numerical values within the same degree of precision are always equal, each, found as in (**146**), being $\frac{m}{n}$ within $\frac{1}{n}$.

$$\text{Then,} \quad \frac{a}{c} = \frac{m}{n} \text{ within } \frac{1}{n}$$

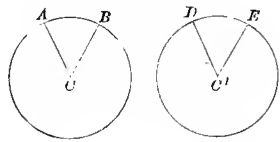
$$\text{and} \quad \frac{b}{d} = \frac{m}{n} \text{ within } \frac{1}{n}.$$

$$\therefore \quad \frac{a}{c} - \frac{b}{d} < \frac{1}{n}.$$

By making n sufficiently great, $\frac{1}{n}$ can be made as small as we please. Hence, $\frac{a}{c} - \frac{b}{d}$ is less than any quantity however small; $\quad \therefore \quad \frac{a}{c} - \frac{b}{d} = 0, \therefore \frac{a}{c} = \frac{b}{d}.$

148. Proposition XXIV.—Theorem.

In the same circle or in equal circles, equal central angles intercept equal arcs of the circumference, and conversely.



Let ACB, $DC'E$, be equal central angles in equal circles, C, C'. Then, the arcs, AB, DE, are equal.

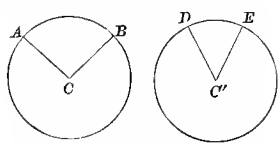
For, place one circle on the other, so that the equal angles coincide. A will coincide with D, and B with E; hence, the arcs, AB, DE, coincide, and are therefore equal.

If the equal central angles are in the same circle, each of the intercepted arcs is equal to the arc intercepted by an equal central angle in an equal circle; hence these arcs are equal to each other.

Conversely, if the arcs, AB, DE, are equal, the angles, ACB, $DC'E$, are equal. For, make the equal circles coincide, also the equal arcs, AB, DE. Then, since the centers coincide, also, A and D, and B and E, AC coincides with DC', and BC with EC'; hence, the angles, ACB, $DC'E$, coincide, and are therefore equal.

149. Proposition XXV.—Theorem.

In the same circle or in equal circles, the greater of two unequal central angles intercepts the greater arc.



Let C, C', be centers of equal circles, and

$$ACB > DC'E;$$

then, $\quad AB > DE.$

Place one circle on the other so that AC and DC'

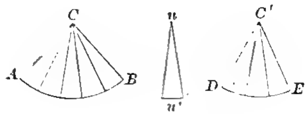
shall coincide. Then, since $DC'E < ACB$, $C'E$ will fall between CA and CB, and E between A and B; hence, $AB > DE$.

Conversely, if $AB > DE$, $ACB > DC'E$. For, place one circle on the other so that AC and DC' shall coincide. Then, since $AB > DE$, E will fall between A and B, and EC' between AC and BC; hence, $ACB > DC'E$.

150. Proposition XXVI.—Theorem.

In the same circle or in equal circles, the ratio of two central angles is equal to the ratio of their intercepted arcs.

1. When the angles are *commensurable:*



Let ACB and $DC'E$ be central angles of equal circles, and suppose these angles have a common unit of measure u which, for example, is contained 5 times in ACB and 3 times in $DC'E$. Then, 5 is the numerical value of ACB, and 3, of $DC'E$.

$$\therefore \quad ACB \quad DC'E :: 5 : 3 \quad (\mathbf{145}).$$

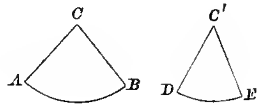
Let radii be drawn dividing ACB into 5 parts, and $DC'E$ into 3 parts, each equal to u. These radii divide AB into 5 parts, and DE into 3 parts, all equal (**148**). Then, 5 is the numerical value of AB, and 3 of DE.

$$\therefore \quad AB : DE :: 5 : 3.$$

Comparing these proportions, we have

$$ACB : DC'E :: AB : DE \quad (\text{ALGEBRA, } \mathbf{317}, 9).$$

2. When the angles are *incommensurable:*



Let ACB and $DC'E$ be incommensurable central angles of equal circles. Suppose the angle $DC'E$ divided into a number n of equal parts, without a remainder. Then, since the angles are incommensurable, ACB will contain a certain number m of these parts, with a remainder less than one part.

Then, $$\frac{ACB}{DC'E} = \frac{m}{n} \text{ within } \frac{1}{n} \quad (\mathbf{146}).$$

The radii, separating these parts, divide DE into n equal parts, without a remainder, and AB into m such parts (**148**), with a remainder less than one part (**149**).

Then, $$\frac{AB}{DE} = \frac{m}{n} \text{ within } \frac{1}{n}.$$

Since the approximate numerical values of the ratios of the angles and arcs, within the same degree of precision, however exact, are always equal, the ratios themselves are equal (**147**).

$$\therefore \quad \frac{ACB}{DC'E} = \frac{AB}{DE}; \text{ that is, } ACB : DC'E :: AB : DE.$$

151. Corollaries.

1. *Assuming a central angle as the unit of measure for angles, and the intercepted arc as the unit of measure for arcs, the numerical value of any central angle is equal to the numerical value of its intercepted arc.*

Let c be any central angle, a its intercepted arc, u the unit of measure for angles, and u' the unit of measure for arcs, u' being the arc intercepted by u.

Then, $$\frac{c}{u} = \frac{a}{u'}, \quad \textbf{(150)}.$$

But $\frac{c}{u}$ is the numerical value of the central angle c, and $\frac{a}{u'}$ is the numerical value of its intercepted arc a (**144,** 8). Hence, the numerical value of a central angle is equal to that of its intercepted arc. On this account, *a central angle is said to be measured by its intercepted arc.*

2. *The ratio of any central angle to four right angles is equal to the ratio of its intercepted arc to the circumference.*

For, four right angles may be regarded as the central angle whose intercepted arc is the circumference.

3. *In the same circle or in equal circles, the ratio of two sectors is equal to the ratio of their arcs.*

The proof is similar to that for Prop. XXVI.

4. *The ratio of a sector to the circle is equal to the ratio of the arc of the sector to the circumference.*

For, the circle may be regarded as a sector whose arc is the circumference.

152. Scholium.

Two diameters perpendicular to each other form four equal central angles, and divide the circumference into four equal arcs (**148**), each of which, called a quadrant, is intercepted by a right angle.

The unit of measure for angles, which is $\frac{1}{90}$ of a right angle, is called a degree, and intercepts $\frac{1}{90}$ of a quadrant, which is the unit of measure for arcs, and is also called a degree.

A degree, whether of angle or arc, is written $1°$.

A minute is $\frac{1}{60}$ of $1°$, and is written $1'$.

A second is $\frac{1}{60}$ of $1'$, and is written $1''$.

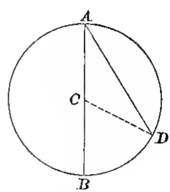
Thus, $10° \, 30' \, 20''$ is read, 10 degrees, 30 minutes, 20 seconds.

The number of degrees in an angle is equal to the number of degrees in its measuring arc.

Equal central angles in unequal circles intercept arcs of the same number of degrees, though differing in length.

153. Proposition XXVII.—Theorem.

An inscribed angle is measured by one-half of its intercepted arc.



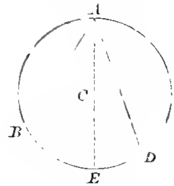
1. Let BAD be an inscribed angle, the center being on the side AB. Draw the radius CD. Then, since BCD is an exterior angle to the triangle ACD, $CAD + CDA = BCD$ (**53,** 1). Since CA, CD, are equal, the triangle ACD is isosceles;

$$\therefore \quad CDA = CAD \ (\mathbf{59}); \quad \therefore \quad 2CAD = BCD;$$

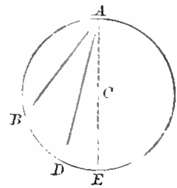
$$\therefore \quad BAD = \tfrac{1}{2}BCD.$$

But the central angle BCD is measured by the arc BD (**151,** 1); hence, the angle BAD is measured by $\frac{1}{2}BD$.

2. Let BAD be an inscribed angle, the center being within the angle. Draw the diameter AE. Then, by the first case, BAE is measured by $\frac{1}{2}BE$, and EAD by $\frac{1}{2}ED$; $\therefore$ $BAE + EAD$, or BAD is measured by $\frac{1}{2}BE + \frac{1}{2}ED$, that is, by $\frac{1}{2}(BE + ED)$, or by $\frac{1}{2}BD$.



3. Let BAD be an inscribed angle, the center being without the angle. Draw the diameter AE. Then, by the first case, BAE is measured by $\frac{1}{2}BE$, and DAE by $\frac{1}{2}DE$; $\therefore$ $BAE - DAE$, or BAD is measured by $\frac{1}{2}BE - \frac{1}{2}DE$, that is, by $\frac{1}{2}(BE - DE)$, or by $\frac{1}{2}BD$.



154. Corollaries.

1. *All angles inscribed in the same segment are equal.*

For, each is measured by one-half of the intercepted arc.

2. *Any angle inscribed in a semicircle is a right angle.*

For, it is measured by one-half of a semi-circumference; that is, by a quadrant which is the measure of a right angle.

3. *Any angle inscribed in a segment greater than a semicircle is an acute angle.* (?)

4. *An angle inscribed in a segment less than a semicircle is an obtuse angle.* (?)

5. *Two inscribed angles are complementary if the half sum of their intercepted arcs is equal to a quadrant, and conversely.* (?)

6. *Two inscribed angles are supplementary if the half sum*

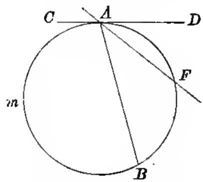
of their intercepted arcs is equal to a semi-circumference, and conversely. (?)

7. *In the same circle or in equal circles, equal inscribed angles intercept equal arcs, and conversely.* (?)

8. *In unequal circles, equal inscribed angles intercept arcs of the same number of degrees, and conversely.* (?)

155. Proposition XXVIII.—Theorem.

An angle formed by a tangent and a chord drawn to the point of tangency is measured by one-half of the intercepted arc.

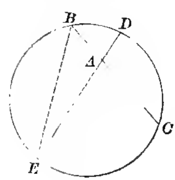


Let CD be a tangent, and AB a chord drawn to the point of tangency. Draw the secant AF, and revolve it about A as a center till the point F coincides with A, when the secant becomes the tangent CD, and the arc BF, the arc BFA.

But in all positions of AF, the inscribed angle BAF is measured by $\frac{1}{2}BF$; hence, BAD is measured by $\frac{1}{2}BFA$.

156. Proposition XXIX.—Theorem.

An angle formed by two intersecting chords is measured by one-half of the sum of the arcs intercepted by its sides and by the sides of its vertical angle.



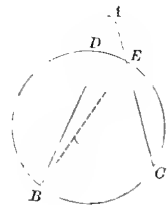
Let BC, DE, be two chords intersecting at A. Drawing BE,

$$EAC = EBC + BED. \quad (?)$$

But EBC is measured by $\frac{1}{2}EC$, and BED by $\frac{1}{2}BD$; $\therefore$ EAC is measured by $\frac{1}{2}EC + \frac{1}{2}BD$; that is, by $\frac{1}{2}(EC + BD)$.

157. Proposition XXX.—Theorem.

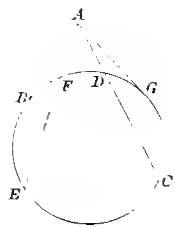
The angle formed by two secants intersecting without the circumference is measured by one-half of the difference of the intercepted arcs.



Let AB, AC, be two secants intersecting at A without the circumference. Drawing BE, $BEC = BAC + ABE$. $\therefore\ BAC = BEC - ABE$. But BEC is measured by $\frac{1}{2}BC$, and ABE by $\frac{1}{2}DE$; $\therefore\ BAC$ is measured by $\frac{1}{2}BC - \frac{1}{2}DE$; that is, by $\frac{1}{2}(BC - DE)$.

158. Corollaries.

1. *The angle formed by a tangent and a secant is measured by one-half of the difference of the intercepted arcs.*



Let AB be a tangent, and AC a secant. Drawing the secant AE between AB and AC, and revolving it about A till it coincides with AB, the points, E, F, will unite at B, EAC will become BAC, also EC will become BC, and FD, BD; but in all positions of AE, EAC is measured by $\frac{1}{2}(EC - FD)$: (?) therefore, BAC is measured by $\frac{1}{2}(BC - BD)$.

2. *The angle formed by two tangents is measured by one-half of the difference of the intercepted arcs.* (?)

159. Exercises.

1. If two circles are tangent externally or internally, and two straight lines be drawn through the point of tangency,

the chords of the intercepted arcs are parallel (**155**), (**154,** 8), (**44,** 2).

2. If two circles have a common tangent, the chords which join the points of tangency and the points in which the straight line through the centers meets the circumferences, on the same side of the centers, are parallel.

3. If two circles are tangent internally, and a chord of the greater is tangent to the less, the line drawn from the point of tangency of the circles to the point of tangency of the chord, bisects the angle formed by the lines drawn from the point of tangency of the circles to the extremities of the chord (**155**), (**44,** 4), (**135**), (**153**).

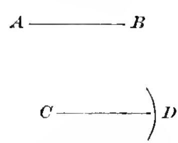
IV. CONSTRUCTIONS.

160. Proposition XXXI.—Problem.

1. *To draw a line equal to a given line.*

2. *To cut off from the greater of two given lines a part equal to the less.*

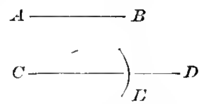
3. *To produce a line till the part produced shall be equal to a given line.*



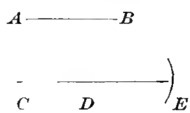
1. Let AB be the given line. Place one point of the dividers on A and extend the other to B. Take the dividers thus extended, place one foot at any point C, about which, as a center, describe an arc with the other foot. Draw a radius CD from C to any point D in the arc. Then, $CD = AB$, since each

is equal to the distance from one foot of the dividers to the other (24, 2).

2. Let AB be the less of the given lines and CD the greater. With a radius equal to AB, found as above, describe about C as a center an arc intersecting CD in E. Then, $CE = AB$.



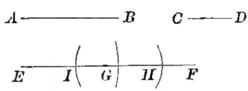
3. Let it be required to produce CD till DE shall be equal to AB. With a radius equal to AB, describe about D as a center an arc on the side of D opposite C. Produce CD by the aid of a ruler till it meets the arc in E. Then, $DE = AB$.



161. Proposition XXXII.—Problem.

To find the sum or difference of two given lines.

Let AB and CD be the given lines. Draw an indefinite line EF. With a radius equal to AB, describe about E as a center an arc intersecting EF in G; and about G as a center, with a radius equal to CD, describe two arcs, one intersecting EF in H, the other in I. From the diagram we have



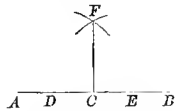
$$EH = EG + GH, \text{ and } EI = EG - GI.$$

$$\therefore \quad EH = AB + CD, \text{ and } EI = AB - CD.$$

162. Proposition XXXIII.—Problem.

To erect a perpendicular to a straight line at a given point in the line.

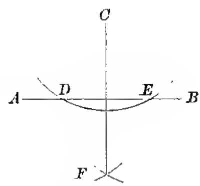
Let AB be the line and C the point. With the dividers lay off from C the equal distances, CD, CE. Opening the dividers a little wider, place one point at D, and with the other describe an arc. Then place one point at E, and with the other describe an arc intersecting the first at F. Draw CF, and it will be perpendicular to AB at C (**70**, 3).



163. Proposition XXXIV.—Problem.

To let fall a perpendicular upon a line from a point without the line.

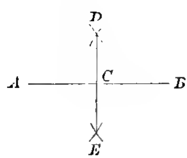
Let AB be the line and C the point. Place one point of the dividers at C, extend the other so that when revolved about C it shall intersect AB in two points, D, E. With D and E as centers, and a radius greater than the half of DE, describe two arcs intersecting at F. Draw CF, and it will be the perpendicular required. (?)



164. Proposition XXXV.—Problem.

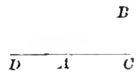
To bisect a given finite straight line.

Let AB be the straight line. With A and B as centers, and a radius greater than the half of AB, describe arcs intersecting in D and E, and draw DE. Then, since each of the points, D, E, is equally distant from A and B, DE is perpendicular to AB at its middle point, and hence, bisects AB.



165. Proposition XXXVI.—Problem.

To construct the supplement of a given angle.



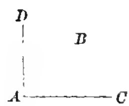
Let BAC be the given angle. Produce CA to D; then, (**31**),

$$DAB + BAC = \text{two right angles};$$

$\therefore$ DAB is the supplement of BAC.

166. Proposition XXXVII.—Problem.

To construct the complement of a given angle.



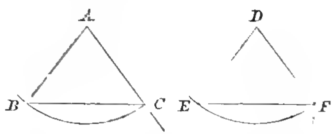
Let BAC be the given angle. Erect AD perpendicular to AC (**162**). Then,

$$DAB + BAC = \text{a right angle}.$$

$\therefore$ DAB is the complement of BAC.

167. Proposition XXXVIII.—Problem.

To construct an angle equal to a given angle.

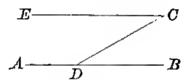


Let A be the given angle. With the vertex A as a center, and any radius AB, describe an arc intersecting the sides of the angle in the points, B, C, and draw the straight line BC. Draw an indefinite straight line DE. With D as a center, and a radius DE equal to AB, describe the indefinite arc EF. With E as a center, and a radius equal to BC, describe an arc intersecting the arc EF in F, and draw the straight line DF. Then,

$$AB = DE,\ AC = DF,\ BC = EF: \ \therefore\ A = D \quad (\mathbf{75}).$$

168. Proposition XXXIX.—Problem.

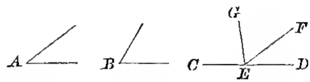
To draw through a given point a line parallel to a given line.



Let C be the given point and AB the given line. Draw a straight line from C to any point of AB, as D. Construct the angle DCE equal to the angle CDB (**167**). Then, CE is parallel to AB (**44**, 2).

169. Proposition XL.—Problem.

Given two angles of a triangle, to find the third.



Let A and B be the given angles. Draw the indefinite straight line CD, and from any point of this line, as E, draw EF, making the angle $DEF = A$; also the line EG, making the angle $FEG = B$ (**167**).

Now, $\quad DEF + FEG + GEC =$ two right angles (**32**, 2).

Hence, GEC is the supplement of $DEF + FEG$ (**27**, 4).

But, $\quad DEF = A, \quad FEG = B,$

$\therefore \quad GEC$ is the supplement of $A + B$.

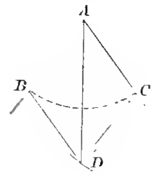
Also, the third angle of the triangle is the supplement of $A + B$. $\therefore$ $GEC =$ the third angle of the triangle.

170. Proposition XLI.—Problem.

To bisect a given angle.

Let BAC be the given angle. With A as a center, and a radius less than either side, describe the arc BC; then,

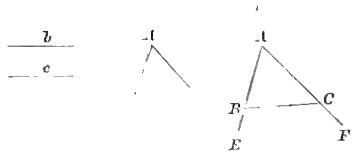
with B and C as centers, and a radius greater than one-half the distance from B to C, describe arcs intersecting in D, and draw AD. Then, AD bisects BAC; for, drawing BD and CD, the triangles, ABD, ACD, are mutually equilateral, and hence mutually equiangular. $\therefore$ the angle $BAD =$ the angle CAD; (?) $\therefore$ AD bisects BAC.



171. Proposition XLII.—Problem.

Given two sides of a triangle and their included angle, to construct the triangle.

Let b, c, be the given sides, and A their included angle. Construct the angle $EAF = A$. On AF lay off $AC = b$, on AE lay off $AB = c$, and draw BC. Then, BAC is the triangle required (55).



172. Proposition XLIII.—Problem.

Given one side and two angles of a triangle, to construct the triangle.

1. When the angles are both adjacent to the given side:

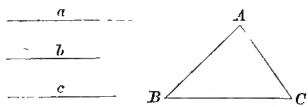
Let a be the given side, B, C, the adjacent angles. Construct the straight line $BC = a$, the angle $CBA = B$, and $BCA = C$ (**160**, 1), (**167**). Then, ABC is the required triangle. (?)

2. When one of the given angles is opposite the given side:

First, find the third angle (**169**), then proceed as above.

173. Proposition XLIV.—Problem.

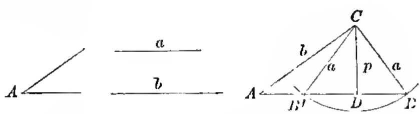
Given the three sides of a triangle, to construct the triangle.



Let a, b, c, be the sides. Draw $BC = a$. With C as a center, and a radius equal to b, describe an arc, and with B as a center, and a radius equal to c, describe an arc intersecting the first arc in A. Then, ABC is the required triangle (**75**).

174. Proposition XLV.—Problem.

Given two sides of a triangle and an angle opposite one of them, to construct the triangle.



Let a and b be the given sides, A the angle opposite a. Construct the angle $BAC = A$, $AC = b$, and p the perpendicular from C to AB.

1. $a < b$.

In this case, A is acute; for, if A were right or obtuse, it would be the greatest angle of the triangle (**53**, 3, 4), and a would be the greatest side (**64**); but this is contrary to the hypothesis that $a < b$.

1st. If $a > p$, there are two solutions. For, with C as a center and a as a radius, an arc can be described inter-

secting the side of the angle A opposite C in two points B and B'; and drawing CB and CB', either triangle ABC, or $AB'C$ having two sides and an angle opposite one of them equal to those given, satisfies the conditions of the problem.

2d. If $a = p$, there is one solution. For, then B and B' unite at D, and the two triangles ABC and $AB'C$ coincide with the right triangle ADC.

3d. If $a < p$, there is no solution. For, a can not reach AD, the construction is impossible, and the triangle imaginary.

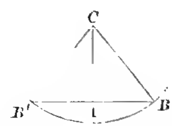
2. $a = b$.

In this case, also, A is acute; for, if A were right or obtuse, since the angles opposite the equal sides are equal (**59**), the triangle would have two right angles or two obtuse angles, which is impossible. (?)

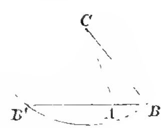
The triangle ABC is real and isosceles, but the triangle $AB'C$ vanishes into the line AC, since B' would coincide with A.

3. $a > b$.

In this case, A may be right or oblique. B and B' fall on opposite sides of A.



1st. If A is right, the triangles, ABC, $AB'C$, are equal, and either satisfies the conditions.

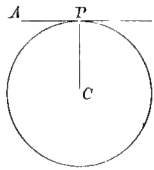


2d. If A is oblique, ABC will satisfy the conditions, but $AB'C$ will not; for, if A is acute, $B'AC$ is obtuse, and if A is obtuse, $B'AC$ is acute.

175. Proposition XLVI.—Problem.

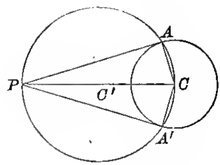
Through a given point to draw a tangent to a circumference.

1. If the given point is on the circumference:



Let P be the given point, and C the center of the given circle. Draw the radius CP, and erect the perpendicular PA; then PA will be tangent (**133**).

2. If the given point is without the circumference:



Let P be the given point, and C the center of the given circle. Draw PC and bisect it at C'. With C' as a center, and CC' as a radius, describe a circumference intersecting the given circumference in A, A'. Draw PA, PA', both of which will be tangent to the circumference of the given circle.

For, drawing the radii, CA, CA', the angles, PAC, $PA'C$, are right angles (**154,** 2); hence, PA, PA', are tangents. (?)

3. If the given point is within the circumference, through it no tangent can be drawn. (?)

176. Corollaries.

1. *The two tangents drawn from a point without a circle to the circumference are equal, and make equal angles with the line drawn from that point to the center.* (?) (**72,** 1).

2. *A line drawn through any point of a tangent, making with the line drawn from that point to the center an angle equal to that made by the tangent, is also a tangent.* (?) (**176,** 1).

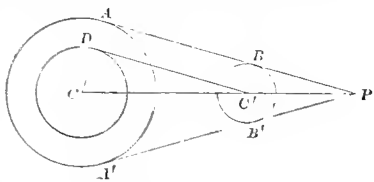
3. *The line drawn from the center to the point of intersection of two tangents bisects the angle formed by the radii drawn to the points of tangency.* (?)

177. Proposition XLVII.—Problem.

To draw a common tangent to two given circles.

1. When the common tangent is *exterior:*

Let C and C' be the centers of the given circles, the radius of the first being the greater.



With C as a center, and a radius CD equal to the difference of the radii of the given circles, describe a circumference.

From C' draw $C'D$ tangent to this circumference (**175**, 2). Draw CD and produce it to A in the given circumference, also, $C'B$ parallel to CA. Draw AB and it will be a common exterior tangent.

For, $CD = CA - C'B$, also, $CD = CA - DA$.

$\therefore \quad C'B = CA - CD$, and $DA = CA - CD$.

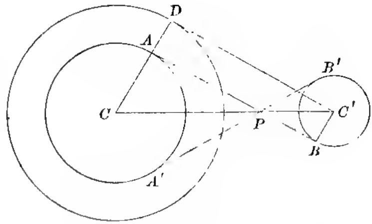
$\therefore \quad C'B = DA$; $\therefore$ $ADC'B$ is a parallelogram (**86**).

But D is a right angle (**133**); $\therefore$ A and B are right angles (**81**). Hence, AB is a tangent to both circumferences. (?)

Produce AB and CC' till they meet in P; from P draw PA', making the angle CPA' equal to the angle CPA; $A'B'P$ is another common exterior tangent (**176**, 2).

2. When the common tangent is *interior:*

With C as a center, and a radius equal to the sum of the radii, describe a circumference.



From C' draw the tangent $C'D$. Draw CD intersecting the given circumference in some point A. Draw $C'B$ parallel to CD and in the opposite direction. Draw AB and it will be a common interior tangent.

For, $CD = CA + C'B$, also, $CD = CA + AD$.

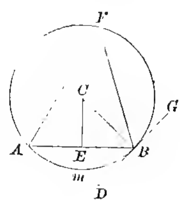
$\therefore \quad C'B = CD - CA$, and $AD = CD - CA$.

$\therefore \quad C'B = AD$; $\therefore$ $ABC'D$ is a parallelogram (**86**).

But D is a right angle; (?) $\therefore$ A and B are right angles; $\therefore$ AB is a common interior tangent. (?)

178. Proposition XLVIII.—Problem.

On a given straight line to describe a segment which shall contain a given angle.



Let AB be the given line, and ABD an angle equal to the given angle. Draw BC perpendicular to BD, and EC perpendicular to AB at its middle point, intersecting BC in C.

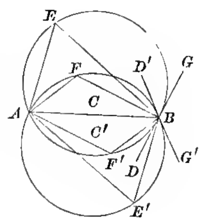
About C as a center, with CB as a radius, describe a circumference; then, ABF is the segment required.

For, BD is a tangent (**133**), and the angle ABD is

measured by $\frac{1}{2}AmB$ (**155**). But any angle AFB, inscribed in the segment ABF, is also measured by $\frac{1}{2}AmB$, and therefore equal to the given angle. Any angle inscribed in the segment ABm is equal to ABG, and is therefore the supplement of ABD.

179. Corollaries.

1. *The locus of the vertices of all the angles whose sides pass through two fixed points, and whose magnitude is equal to that of the angle formed by the common chord of two equal circumferences passing through these points with the interior portion of the tangent to either circumference, at either of these points, is the exterior arcs of the circumferences.*



Let C, C', be the centers of equal circumferences passing through the given fixed points, A, B.

The common chord AB of the equal circles subtends equal arcs (**126**).

$\therefore$ $AEB = AE'B = ABD = ABD'$ (**153**), (**155**).

Now, any angle whose sides pass through the points, A, B, respectively, and whose vertex is within either arc, AEB, $AE'B$, is within some angle whose vertex is in one of these arcs, and is therefore greater than that angle (**54**). For like reason, any angle whose sides pass through A, B, and whose vertex is without the arcs, AEB, $AE'B$, is less than the angle AEB.

Hence, $AEBE'A$ is the specified locus.

2. *The locus of the vertices of all the angles whose sides pass through two fixed points, and whose magnitude is equal to that of the angle formed by the common chord of two equal circum-*

ferences passing through these points with the exterior portion of the tangent to either circumference, at either of these points, is the interior arcs of the circumferences. (?)

3. *The locus of the vertices of all the right angles whose sides pass through two fixed points, is the circumference whose diameter is the straight line terminated by the fixed points.* (?)

180. Exercises.

1. Construct a right triangle; given, first, the hypotenuse and a side; second, the hypotenuse and an acute angle.

2. Trisect a right angle (**53,** 10).

3. From two points on the same side or on opposite sides of a line, draw two equal lines which shall meet in the given line.

4. From two points on the same side of a line, draw two lines which shall meet in the given line and make with it equal angles (**76,** 5).

5. Construct a square, given the difference between its diagonal and side.

6. Draw a tangent to a circumference; first, parallel to a given line; second, perpendicular to a given line.

7. Determine a point in the prolongation of a diameter such that a tangent drawn from it to the circumference shall be equal to a given line.

8. With a given radius describe a circumference passing through a given point and tangent to a given straight line (**50,** 1), (**117,** 5).

9. Describe a circumference which shall pass through two given points and have its center in a given straight line.

10. With a given radius describe a circumference tangent to a given circumference and to a given straight line.

11. Describe a circumference tangent to a given straight line at a given point, such that the tangents drawn to it from two given points in the straight line may be parallel.

12. With a given radius describe a circumference tangent to a given straight line, such that the tangents drawn to it from two given points in the straight line may be parallel.

13. To construct a triangle, given the base, the vertical angle, and the point of intersection of the base with the perpendicular to the base from the vertex of the opposite angle (**178**).

14. To construct a triangle, given the base, the altitude, and the vertical angle.

15. To construct a triangle, given the base, the medial line to the base, and the vertical angle.

16. Through a given point draw a straight line so that the part intercepted by a given circumference shall be equal to a given straight line.

V. INSCRIBED AND CIRCUMSCRIBED POLYGONS.

181. Definitions.

1. An **inscribed polygon** is a polygon whose vertices are all in the circumference. In this case, the circle is circumscribed about the polygon.

2. A **circumscribed polygon** is a polygon whose sides are all tangent to the circumference. In this case, the circle is inscribed in the polygon.

3. An **inscriptible polygon** is a polygon which can be inscribed in a circle.

4. A **circumscriptible polygon** is a polygon which can be circumscribed about a circle.

182. Proposition XLIX.—Theorem.

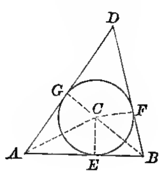
Any triangle is inscriptible.

For, a circumference can be made to pass through the vertices (**130**).

183. Proposition L.—Theorem.

Any triangle is circumscriptible.

Let ABD be a triangle. Bisect any two of its angles, as A and B, by straight lines meeting in C. From C let fall the perpendiculars, CE, CF, CG, on the three sides of the triangle. Then, these perpendiculars are equal. For, the right triangles, ACE, ACG, are equal (**72,** 2); $\therefore$ $CE = CG$ (**56,** 2).



In like manner, it is proved that $CE = CF$.

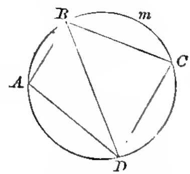
Hence, the circumference having C for a center and CE for a radius will pass through the points E, F, G.

The sides of the triangle are tangents, since each is perpendicular to a radius at its extremity (**133**).

Hence, the triangle is circumscriptible (**181,** 4, 2).

184. Proposition LI.—Theorem.

If a quadrilateral is inscriptible, its opposite angles are supplementary; and conversely, if the opposite angles of a quadrilateral are supplementary, the quadrilateral is inscriptible.



Let the quadrilateral $ABCD$ be inscribed in a circle. Then, its opposite angles are supplementary.

For, the angle A is measured by one-half the arc BCD, and the angle C by one-half the arc DAB (**153**); hence, $A + C$ is measured by one-half the sum of the arcs BCD and DAB; that is, by a semi-circumference; $\therefore$ A and C are supplementary (**154,** 6); $\therefore$ ABC and ADC are supplementary (**79**).

Conversely, if A and C are supplementary, $ABCD$ is inscriptible.

Describe a circumference through the points, B, A, D, and draw the chord BD.

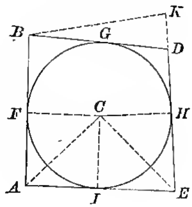
Now, any angle inscribed in the segment BmD is the supplement of the angle A. (?)

If the vertex C were within the arc BmD, the angle C would be greater than the supplement of the angle A (**54**); if the vertex C were without the arc BmD, the angle C would be less than the supplement of the angle A; (?) but both of these results are contrary to the hypothesis; hence, the vertex C can be neither within nor without the arc, and must, therefore, be on the arc BCD; $\therefore$ $ABCD$ is inscriptible.

185. Proposition LII.—Theorem.

If a quadrilateral is circumscriptible, the sum of two opposite sides is equal to the sum of the other two sides; and conversely,

if the sum of two opposite sides of a quadrilateral is equal to the sum of the other two sides, the quadrilateral is circumscriptible.



Let the circumscriptible quadrilateral $ABDE$ be circumscribed about a circle, F, G, H, I, being the points of tangency.

Now, $AF = AI$, $BF = BG$,

$DH = DG$, $EH = EI$ (**176, 1**).

$$\therefore \quad AF + BF + DH + EH = AI + BG + DG + EI.$$

$$\therefore \quad AB + DE = AE + BD.$$

Conversely, if $AB + DE = AE + BD$, then, $ABDE$ is circumscriptible.

There must be two consecutive angles, for example, A and E, whose sum does not exceed two right angles, since the sum of the four angles is equal to four right angles.

Let AC and EC be the bisectors of the angles A and E. The sum of the angles, CAE, CEA, is less than two right angles; (?) therefore, AC and EC intersect in some point C (**47**).

The perpendiculars let fall from C to the three sides, AB, AE, ED, are equal (**105**). If, therefore, with C as a center, and one of these perpendiculars as a radius, a circumference be described, it will be tangent to the sides AB, AE, ED.

It is now to be proved that this circumference is tangent to the fourth side BD.

If BD is not tangent, let BK be tangent; then, BK will intersect ED, or ED produced in some point as K, since the sum of the angles, A, E, does not exceed two right angles, and since BK, if tangent, must fall between AB

and ED. If BK is tangent, $ABKE$ is a circumscribed quadrilateral; then,

$$AB + KE = BK + AE, \text{ but } AB + DE = BD + AE.$$

$$\therefore\ KE - DE = BK - BD, \quad \text{or} \quad KD = BK - BD.$$

That is, one side of a triangle is equal to the difference of the other sides, which is impossible (**66**); hence, the supposition that BD is not a tangent, which led to this impossibility, is false (**15**, 4). Therefore, BD is tangent, and $ABDE$ is circumscriptible.

186. Exercises.

1. Circumscribe a circle about a given rectangle.

2. Inscribe a circle in a given rhombus.

3. In a given circle inscribe a triangle whose angles are respectively equal to the angles of a given triangle.

4. About a given circle circumscribe a triangle whose angles are respectively equal to the angles of a given triangle.

5. Circumscribe a circle about a triangle, and prove by (**153**) that the sum of the three angles is equal to two right angles.

6. Prove by (**127**), (**153**), that the greatest angle of a triangle is opposite the greatest side; that, in an isosceles triangle, the angles opposite the equal sides are equal; and that an equilateral triangle is equiangular.

7. In a right triangle, the sum of the hypotenuse and diameter of the inscribed circle is equal to the sum of the other sides.

8. The central angles subtended by two opposite sides of a circumscriptible quadrilateral are supplementary.

VI. SYMMETRY.—SUPPLEMENTARY.

187. Definitions.

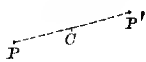
1. Two objects are *symmetrical* when every point of either has a corresponding point in the other on the opposite side of a point, line, or plane, taken as an object of reference, and at an equal distance from it.

2. A point taken as the object of reference is called the *center of symmetry.*

3. A line taken as the object of reference is called the *axis of symmetry.*

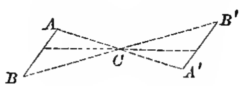
4. A plane taken as the object of reference is called the *plane of symmetry.*

5. Two points are symmetrical with respect to a *center* when the center bisects the straight line terminated by these points. Thus, P, P', are symmetrical with respect to C as a center, if C bisects PP'.

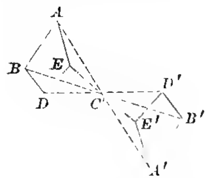


6. The distance of either of two symmetrical points from the center of symmetry is called the *radius of symmetry.* Thus, either CP or CP' is the radius of symmetry.

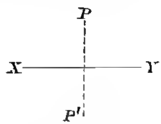
7. Two figures are symmetrical with respect to a center when every point of either has its symmetrical point in the other with respect to that center. Thus, the lines AB, $A'B'$, are symmetrical with respect to the center C, if every point of either has its symmetrical point in the other, with respect to C as a center.



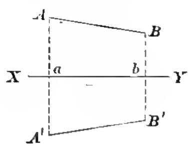
Also, the polygons, $ABDE$, $A'B'D'E'$, are symmetrical with respect to C as a center, if every point in the perimeter of either has its symmetrical point in the perimeter of the other, with respect to C as a center.



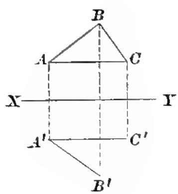
8. Two points are symmetrical with respect to an axis when the axis bisects at right angles the straight line terminated by these points. Thus, P, P', are symmetrical with respect to the axis XY, if XY bisects PP' at right angles.



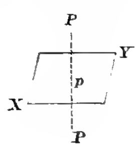
9. Two figures are symmetrical with respect to an axis when every point of either has its symmetrical point in the other, with respect to that axis. Thus, the lines, AB, $A'B'$, are symmetrical with respect to XY, if every point of either has its symmetrical point in the other, with respect to XY as an axis.



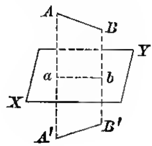
Also, the triangles, ABC, $A'B'C'$, are symmetrical with respect to the axis XY, if every point of the perimeter of either has its symmetrical point in the perimeter of the other, with respect to XY as an axis.



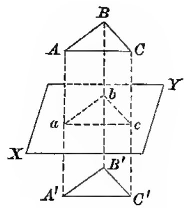
10. Two points are symmetrical with respect to a plane when the plane bisects at right angles the straight line terminated by these points. Thus, P, P', are symmetrical with respect to XY, if XY bisects PP' at right angles.



11. Two figures are symmetrical with respect to a plane, when every point of either has its symmetrical point in the other with respect to that plane. Thus, AB, $A'B'$, are symmetrical with respect to XY, if every point of either has its symmetrical point in the other with respect to XY as a plane of symmetry.

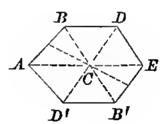


Also, the triangles, ABC, $A'B'C'$, are symmetrical with respect to the plane XY, if every point in the perimeter of the one has its symmetrical point in the perimeter of the other with respect to XY as a plane of symmetry.

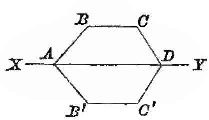


12. Symmetrical points and lines in two symmetrical figures are called *homologous*.

13. A single figure is symmetrical, when it has a center, axis, or plane, with respect to which every point of its perimeter or surface has its symmetrical point in the perimeter or surface. Thus, $ABDEB'D'$ is symmetrical with respect to the center C, if C bisects every straight line through it terminated by the perimeter.



Also, $ABCDC'B'$ is symmetrical with respect to the axis XY, if XY so divides it that the portions $ABCD$, $AB'C'D$, are symmetrical with respect to XY as the axis of symmetry.



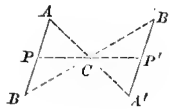
14. The straight line drawn through the center of a single symmetrical figure, and terminated by the perimeter if the

figure is plane, or by the surface if the figure is solid, is called a *diameter*.

15.* Symmetry is *right* or *oblique* according as the axis or plane of symmetry bisects, at right or oblique angles, the lines joining homologous points.

188. Proposition LIII.—Theorem.

The symmetrical of a straight line with respect to a center is an equal and parallel straight line.



Let AB be a straight line, and C the center of symmetry. Draw AC, and produce it till CA' is equal to AC. Then, A' is the symmetrical of A.

Likewise, find B' the symmetrical of B, and draw $A'B'$.

The triangles, ACB, $A'CB'$, are equal (**55**); $\therefore A'B' = AB$, and the angle A' = the angle A; hence, $A'B'$ is parallel to AB (**44**, 2).

From P, any point of AB, draw PC, and produce it to P' in $A'B'$. Then, the triangles, ACP, $A'CP'$, are equal (**57**); $\therefore CP' = CP$; therefore, P' is the symmetrical of P; but P is any point of AB; hence, every point of AB has its symmetrical in $A'B'$; therefore, $A'B'$ is the symmetrical of AB.

189. Corollaries.

1. *Two lines symmetrical with respect to a center lie in opposite directions from their symmetrical extremities as origins.*

2. *Two equal parallel lines are symmetrical with respect to a center.*

3. *If the extremities of one line are respectively the symmetricals of the extremities of another line, with respect to the same center, the two lines are symmetrical with respect to that center.*

190. Exercises.

1. The symmetrical of a polygon with respect to a center is an equal polygon.

2. A point can be made to coincide with its symmetrical by revolving its radius of symmetry in the same plane through two right angles.

3. How can a figure be made to coincide with its symmetrical with respect to a center?

4. The symmetrical of a straight line with respect to an axis is an equal straight line.

5. If a straight line is perpendicular or parallel to the axis of symmetry, its symmetrical is perpendicular or parallel to the axis.

6. If a straight line is oblique to the axis of symmetry, its symmetrical is oblique to the axis, and the two symmetricals intersect the axis at the same point and make equal angles with it.

7. If two straight lines intersect, their symmetricals with respect to an axis intersect, and the points of intersection are symmetrical.

8. Construct the symmetrical of a polygon with respect to an axis, and prove that it is equal to the given polygon.

9. If a figure is symmetrical with respect to two axes at right angles, it is symmetrical with respect to their intersection as a center.

BOOK III.

I. AREA AND EQUIVALENCY.

191. Definitions.

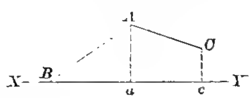
1. A **superficial unit** is an assumed unit of measure for surfaces (**144,** 4). It is usually the square whose side is a linear unit.

2. The **area** of a surface is its numerical value (**144,** 8).

3. **Equivalent surfaces** are those whose areas are equal.

4. The **projection** of a point upon a line is the foot of the perpendicular drawn from the point to the line. If the point is on the line, it is its own projection.

5. The projection of a finite straight line upon a given line is the distance between the projections of its extremities. Thus, a is the projection of A, B is its own projection, ac is the projection of AC, aB of AB.

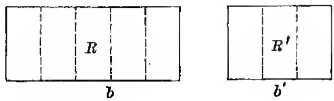


192. Proposition I.—Theorem.

Rectangles having equal altitudes are proportional to their bases.

Let b, b', denote the bases of two rectangles, R, R', having equal altitudes.

1. Suppose the bases are commensurable, and that the common unit of measure is contained, for example, 5 times in b and 3 times in b'.



Then, $b : b' :: 5 : 3$ (**145**).

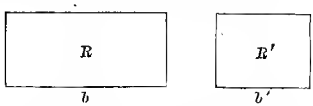
Now, b can be divided into 5 parts and b' into 3 parts, each equal to the common unit.

If perpendiculars be erected to the bases at the several points of division, R will be divided into 5 rectangles and R' into 3 rectangles, all equal, since they have equal bases and equal altitudes (**83**, 3).

$$\therefore \quad R : R' :: 5 : 3. \quad (?)$$

$$\therefore \quad R : R' :: b : b' \quad (\text{ALG., } \mathbf{317},\ 9).$$

2. Suppose the bases are incommensurable, and that b' is divided into n equal parts without a remainder.



Since b, b', are incommensurable, b will contain a certain number m of these parts, with a remainder less than one part.

Then, $$\frac{b}{b'} = \frac{m}{n} \text{ within } \frac{1}{n} \quad (\mathbf{146}).$$

The perpendiculars to the bases at the points of division divide R' into n rectangles without a remainder, and R into

m rectangles with a remainder. These resulting rectangles are all equal, except the remainder, which is less than one of these rectangles.

$$\therefore \quad \frac{R}{R'} = \frac{m}{n} \text{ within } \frac{1}{n} \quad \textbf{(146)}.$$

$$\therefore \quad \frac{R}{R'} = \frac{b}{b'}, \text{ or } R : R' :: b : b' \quad \textbf{(147)}.$$

193. Corollary.

Rectangles having equal bases are proportional to their altitudes.

For, the bases may be taken as altitudes and the altitudes as bases.

194. Scholiums.

1. The term *rectangles*, in the proposition and corollary, is the usual abbreviated expression for the *areas of rectangles.*

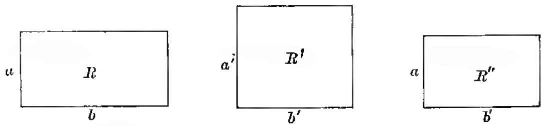
2. The *rectangle of two lines* is the rectangle having one of these lines for its base and the other for its altitude.

3. By the *product of two lines* is to be understood the *product of their numerical values.* If the lines are equal, their rectangle is a *square,* and their product a *square number.*

195. Proposition II.—Theorem.

Any two rectangles are proportional to the products of their bases by their altitudes.

Let R, R', denote two rectangles, b, b', their bases, and a, a', their altitudes.



Construct a third rectangle, R'', whose altitude is a, the altitude of R, and whose base is b', the base of R'.

Then, $\quad R : R'' :: b : b' \quad$ (**192**).

Also, $\quad R'' : R' :: a : a' \quad$ (**193**).

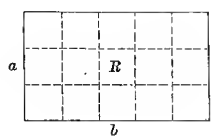
Taking the product of the corresponding terms of these proportions, and omitting R'', the common factor of the first couplet, we shall have

$$R : R' :: ab : a'b' \quad (\text{Alg., } \mathbf{317}, 17, 13).$$

196. Proposition III.—Theorem.

The area of a rectangle is equal to the product of its base by its altitude.

1. When the base and altitude are *commensurable:*



Let R denote a rectangle, b the numerical value of its base, and a of its altitude, referred to a common linear unit of measure.

The base can be divided into b parts and the altitude into a parts, each equal to the linear unit.

Perpendiculars to the base and altitude at the points of

division divide the rectangle into superficial units, of which there are a rows of b units each.

Since there are b superficial units in one row, in a rows there are a times b, or ab superficial units.

$$\therefore \quad R = ab.$$

2. When the base and altitude are *incommensurable:*

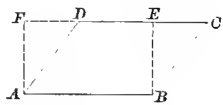
Let R, R', denote two rectangles, b, b', their bases, a, a', their altitudes, a, b, being commensurable, a', b', incommensurable.

Then, $\quad R : R' :: ab : a'b' \quad$ (**195**).

But, $\quad R = ab, \therefore R' = a'b' \quad$ (ALG., **317**, 2).

197. Proposition IV.—Theorem.

The area of a parallelogram is equal to the product of its base by its altitude.



Let P denote the area of the parallelogram $ABCD$, b its base, and a its altitude.

Erect the perpendiculars AF, BE, to the base. Produce CD to F, forming the rectangle $ABEF$, which denote by R.

The right triangles, AFD, BEC, are equal (**82**). (**72**, 1). Taking these triangles, in succession, from the whole figure, we have the parallelogram $ABCD$ equivalent to the rectangle $ABEF$. $\therefore \; P = R$; but $R = ab$, $\therefore \; P = ab$.

198. Corollaries.

1. *Any two parallelograms are proportional to the products of their bases by their altitudes.*

For, let P, P', denote two parallelograms, b, b', their bases, a, a', their altitudes.

Then, $\quad P = ab$, and $P' = a'b'$.

$\therefore \quad P : P' :: ab : a'b'$.

2. *Parallelograms having equal altitudes are proportional to their bases.*

For, $\quad P : P' :: ab : a'b' \quad$ (**198,** 1).

If $a' = a$, $\quad P : P' :: b : b' \quad$ (ALG., **317,** 13).

3. *Parallelograms having equal bases are proportional to their altitudes.*

For, $\quad P : P' :: ab : a'b'$.

If $b' = b$, $\quad P : P' :: a : a'$.

4. *Parallelograms having equal bases and equal altitudes are equivalent.*

For, $\quad P : P' :: ab : a'b'$.

If $b = b'$ and $a = a'$, $ab = a'b'$; $\therefore \; P = P'$.

5. *Parallelograms having bases reciprocally proportional to their altitudes are equivalent.*

For, $\quad P : P' :: ab : a'b'$.

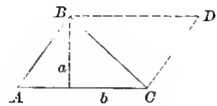
If $\quad b : b' :: a' : a$.

Then, $\quad ab = a'b'$; $\therefore \; P = P'$.

199. Proposition V.—Theorem.

The area of a triangle is equal to one-half the product of its base by its altitude.

Let T denote the area of the triangle ABC, b the numerical measure of its base, and a of its altitude. Draw BD parallel to AC, and CD parallel to AB, forming the parallelogram $ABDC$, and denote the area of this parallelogram by P.



Now, the triangle ABC is one-half the parallelogram $ABDC$ (**83**, 1); that is, $T = \frac{1}{2}P$.

But, $$P = ab; \quad \therefore \quad T = \tfrac{1}{2}ab.$$

200. Corollaries.

1. *A triangle is equivalent to one-half of any parallelogram having an equal base and an equal altitude.* (?)

2. *Any two triangles are proportional to the products of their bases by their altitudes.* (?)

3. *Triangles having equal altitudes are proportional to their bases.* (?)

4. *Triangles having equal bases are proportional to their altitudes.* (?)

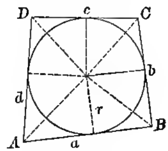
5. *Triangles having equal bases and equal altitudes are equivalent.* (?)

6. *Triangles having bases inversely proportional to their altitudes are equivalent.* (?)

201. Proposition VI.—Theorem.

The area of any circumscriptible polygon is equal to one-half the product of its perimeter by the radius of the inscribed circle.

Let P denote the area of the polygon $ABCD$; a, b, c, d, respectively, the sides; p, the perimeter; and r, the radius of the inscribed circle.



Lines drawn from the center of the inscribed circle to the vertices divide the polygon into triangles having r for their common altitude and a, b, c, d, respectively, for their bases.

The area of each triangle is equal to one-half the product of its base by its altitude, and the area of the polygon is the sum of the areas of the triangles.

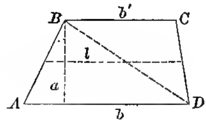
$$\therefore \qquad P = \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr + \tfrac{1}{2}dr,$$

$$\text{or} \qquad P = \tfrac{1}{2}(a + b + c + d)\,r = \tfrac{1}{2}pr.$$

202. Proposition VII.—Theorem.

The area of a trapezoid is equal to one-half the product of its altitude by the sum of its bases.

Let T denote the area of the trapezoid $ABCD$, b the numerical measure of its lower base, b' of its upper base, and a of its altitude.



The diagonal BD divides the trapezoid into two triangles, ABD, BDC, denoted by T', T''. ABD has b for its base and a for its altitude, and BDC has b' for its base and a for its altitude, since a is equal to the perpendicular distance from D to the upper base, or to the upper base produced.

$$\text{Now,} \qquad T = T' + T''.$$

$$\text{But,} \qquad T' = \tfrac{1}{2}ab, \quad \text{and} \quad T'' = \tfrac{1}{2}ab'.$$

$$\therefore \qquad T = \tfrac{1}{2}ab + \tfrac{1}{2}ab' = \tfrac{1}{2}a(b + b').$$

203. Corollary.

The area of a trapezoid is equal to the product of its altitude and the line joining the middle points of its non-parallel sides.

For, let l be the line joining the middle points of AB and CD.

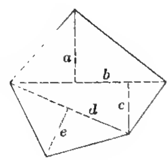
Then, $l = \frac{1}{2}(b + b')$ (**104,** 3).

But, $T = \frac{1}{2}a(b + b') = a \times \frac{1}{2}(b + b')$, $\therefore$ $T = al$.

204. Scholiums.

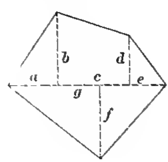
1. The area of a polygon can be found thus:

Draw diagonals from one vertex to the opposite vertices, thus dividing the polygon into triangles; take the diagonals for bases; from the vertices draw perpendiculars to the diagonals, for the altitudes; measure the bases and altitudes; then, $P = \frac{1}{2}ab + \frac{1}{2}cb + \frac{1}{2}ed$.



2. The area of a polygon can also be found thus:

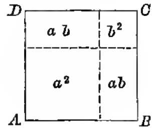
Draw one diagonal, and perpendiculars to this diagonal from the vertices, thus dividing the polygon into triangles and trapezoids; measure the bases and altitudes of the triangles and trapezoids; then, $P = \frac{1}{2}ab + \frac{1}{2}c(b + d) + \frac{1}{2}de + \frac{1}{2}fg$.



205. Proposition VIII.—Theorem.

The square of the sum of two lines is equivalent to the sum of their squares plus twice their rectangle.

Let each side of the square $ABCD$ be the sum of the lines a and b.



The square has $a + b$ for its side and $(a + b)^2$ for its area.

By joining the common extremities of a and b in the opposite sides of the square by straight lines, the whole square will be divided into two squares and two rectangles. (?)

One square has a for its side and a^2 for its area, and the other square has b for its side and b^2 for its area. (?)

Each rectangle has a for one side, b for the other, and ab for its area. (?) Then, $2ab =$ the area of the two rectangles.

$$\therefore \quad (a + b)^2 = a^2 + b^2 + 2ab.$$

206. Corollaries.

1. *The square of twice a line is four times the square of the line.*

For, if $b = a$, the formula of the last article becomes

$$(2a)^2 = a^2 + a^2 + 2a^2 = 4a^2.$$

This is also evident from the diagram; for, if $b = a$, $b^2 = a^2$; each of the rectangles ab becomes a^2; and, since the side is $2a$,

$$(2a)^2 = a^2 + a^2 + a^2 + a^2 = 4a^2.$$

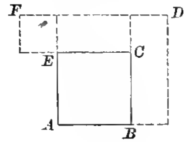
2. *The square of three times a line is nine times the square of the line.* (?)

3. *The square of* n *times a line is* n^2 *times the square of the line.* (?)

207. Proposition IX.—Theorem.

The square of the difference of two lines is equivalent to the sum of their squares minus twice their rectangle.

Let each side of the square AD be a, one side of each of the rectangles, BD, CF, be a, and the other b. Then, each side of AC is $a - b$, and each side of EF is b.



Then, $AC = (a - b)^2$, $AD = a^2$,

$EF = b^2$, $BD = ab$, $CF = ab$.

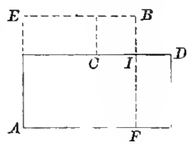
But, $AC = AD + EF - BD - CF$.

$\therefore \quad (a - b)^2 = a^2 + b^2 - 2ab.$

208. Proposition X.—Theorem.

The rectangle of the sum and difference of two lines is equivalent to the difference of their squares.

Let each side of the square AB be a, each side of the square CB be b, the base of the rectangle AD be $a + b$, the altitude, $a - b$. Then, $CE = FD$, since each has $a - b$ for one side and b for the other.



Now, $AD = AI + FD = AI + CE = AB - CB$.

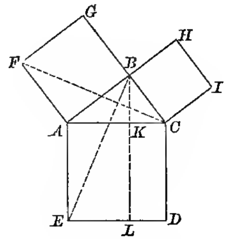
But, $AD = (a + b)(a - b)$, $AB = a^2$, $CB = b^2$.

$$\therefore \quad (a + b)(a - b) = a^2 - b^2.$$

209. Proposition XI.—Theorem.

The square of the hypotenuse of a right triangle is equivalent to the sum of the squares of the other sides.

Let the triangle ABC be right angled at B. Then, AD, the square of AC, is equivalent to $AG + CH$, the sum of the squares of AB and BC.



From B draw BK perpendicular to AC, and produce it to L in ED. Then, KL is parallel to AE (**41**), and AL is a rectangle. Draw BE and FC.

Since ABG and ABC are right angles, GBC is a straight line; and for a similar reason ABH is a straight line (**35**).

The triangles, BAE, FAC, are equal; for, AB is equal to AF, since they are sides of the same square; AE is equal to AC for a similar reason; and the angles BAE and FAC are equal, since each is the sum of the angle BAC and a right angle (**55**).

The rectangle AL is equivalent to twice the triangle BAE; for, it has the same base AE, and, since the vertex B is in LK produced, it has an equal altitude (**200**, 1).

The square AG is equivalent to twice the triangle FAC; for, it has the same base AF, and, since the vertex C is in GB produced, it has an equal altitude.

But the triangle BAE is equal to the triangle FAC; hence, the rectangle AL is equivalent to the square AG.

In like manner, it can be proved that the rectangle KD is equivalent to the square CH.

But the sum of the rectangles AL and KD is the square AD; hence, the square AD is equivalent to the sum of the squares AG and CH; that is, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$.

210. Corollaries.

1. *The square of either side adjacent to the right angle is*

equivalent to the square of the hypotenuse minus the square of the other side.

For, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$, $\therefore \begin{cases} \overline{AB}^2 = \overline{AC}^2 - \overline{BC}^2. \\ \overline{BC}^2 = \overline{AC}^2 - \overline{AB}^2. \end{cases}$

2. *The square of the hypotenuse is to the square of either of the other sides as the hypotenuse is to the projection of that side on the hypotenuse.*

For, $AD : AL :: AC : AK$ (**192**).

Also, $AD : KD :: AC : KC$.

But, $AD = \overline{AC}^2$, $AL = \overline{AB}^2$, $KD = \overline{BC}^2$.

$\therefore$ $\overline{AC}^2 : \overline{AB}^2 :: AC : AK$.

Also, $\overline{AC}^2 : \overline{BC}^2 :: AC : KC$.

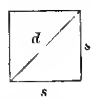
3. *The squares of the sides adjacent to the right angle are proportional to their projections on the hypotenuse.*

For, $AL : KD :: AK : KC$ (**192**).

But, $AL = \overline{AB}^2$, $KD = \overline{BC}^2$.

$\therefore$ $\overline{AB}^2 : \overline{BC}^2 :: AK : KC$.

4. *The square of a diagonal of a square is double the square.*



Let s be one side of the square, and d the diagonal. Then,

$$d^2 = s^2 + s^2 = 2s^2 \quad (\mathbf{209}).$$

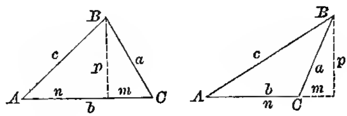
5. *The diagonal of a square is to the side as the square root of two is to one.*

For, $d^2 = 2s^2$. $\therefore$ $d^2 : s^2 :: 2 : 1$. (?)

Hence, $d : s :: \sqrt{2} : 1$. (?)

211. Proposition XII.—Theorem.

In any triangle, the square of a side opposite an acute angle is equivalent to the sum of the squares of the other sides, minus twice the rectangle of one of these sides and the projection of the other on that side.



Let the triangle ABC be acute angled at A; a, b, c, respectively, the sides opposite the angles A, B, C; m, n, respectively, the projections of a and c on b, or on b produced; p, the perpendicular from B to b, or to b produced.

According as the projection of B falls on b, or on b produced, we have

$$m = b - n, \quad \text{or} \quad m = n - b.$$

In either case, by squaring, we have

$$m^2 = b^2 + n^2 - 2bn \quad \textbf{(207)}.$$

Adding p^2 to both members, we have

$$m^2 + p^2 = b^2 + n^2 + p^2 - 2bn.$$

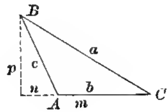
But, $\quad m^2 + p^2 = a^2, \; n^2 + p^2 = c^2 \quad \textbf{(209)}.$

$$\therefore \quad a^2 = b^2 + c^2 - 2bn.$$

212. Proposition XIII.—Theorem.

In an obtuse triangle, the square of the side opposite the obtuse angle is equivalent to the sum of the squares of the other sides, plus twice the rectangle of one of these sides and the projection of the other on that side.

Let the triangle ABC be obtuse angled at A; a, b, c, respectively, the sides opposite the angles A, B, C; and m, n, respectively, the projections of a and c on b and b produced; p, the perpendicular from B to b produced.



Now, $\quad m = b + n.$

$\therefore \quad m^2 = b^2 + n^2 + 2bn \quad$ **(205)**.

$\therefore \quad m^2 + p^2 = b^2 + n^2 + p^2 + 2bn.$

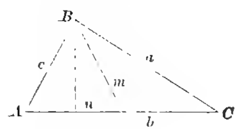
$\therefore \quad a^2 = b^2 + c^2 + 2bn.$

213. Proposition XIV.—Theorem.

1. *In any triangle, the sum of the squares of any two sides is equivalent to twice the square of half the third side, plus twice the square of the medial line to that side.*

2. *The difference of the squares of any two sides is equivalent to twice the rectangle of the third side and the projection, on that side, of the medial line to that side.*

Let ABC be a triangle; a, b, c, respectively, the sides opposite A, B, C; m, the medial line to b; n, the projection of m on b.



Then, $\quad a^2 = \left(\frac{b}{2}\right)^2 + m^2 + bn \quad$ **(212)**.

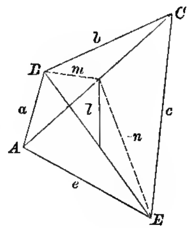
Also, $\quad c^2 = \left(\frac{b}{2}\right)^2 + m^2 - bn \quad$ **(211)**.

$\therefore \quad a^2 + c^2 = 2\left(\frac{b}{2}\right)^2 + 2m^2.$

Also, $\quad a^2 - c^2 = 2bn.$

214. Corollaries.

1. *The sum of the squares of the four sides of any quadrilateral is equivalent to the sum of the squares of the diagonals, plus four times the square of the line joining the middle points of the diagonals.*



Let $ABCE$ be a quadrilateral whose sides are denoted as in the diagram; d, d', respectively, the diagonals, AC, BE; m, n, respectively, medial lines from B and E to the diagonal AC; and l the line joining the middle points of the diagonals.

Then, $$a^2 + b^2 = 2\left(\frac{d}{2}\right)^2 + 2m^2 \quad (\mathbf{213},\ 1).$$

Also, $$c^2 + e^2 = 2\left(\frac{d}{2}\right)^2 + 2n^2.$$

$$\therefore \quad a^2 + b^2 + c^2 + e^2 = 4\left(\frac{d}{2}\right)^2 + 2(m^2 + n^2).$$

But, $$m^2 + n^2 = 2\left(\frac{d'}{2}\right)^2 + 2l^2. \quad (?)$$

$$\therefore \quad a^2 + b^2 + c^2 + e^2 = 4\left(\frac{d}{2}\right)^2 + 4\left(\frac{d'}{2}\right)^2 + 4l^2.$$

$$\therefore \quad a^2 + b^2 + c^2 + e^2 = d^2 + d'^2 + 4l^2.$$

2. *The sum of the squares of the four sides of a parallelogram is equivalent to the sum of the squares of the diagonals.*

For, the diagonals of a parallelogram bisect each other, and l in the formula of Cor. 1 becomes 0.

$$\therefore \quad a^2 + b^2 + c^2 + e^2 = d^2 + d'^2.$$

215. Exercises.

1. The difference of the squares of any two sides of a triangle is equivalent to the difference of the squares of the projections of these sides on the third side (**209**).

2. In any quadrilateral, the sum of the squares of the diagonals is equivalent to twice the sum of the squares of the straight lines joining the middle points of the opposite sides (**114**, 1), (**214**, 2).

3. In an isosceles triangle, the square of one of the equal sides is equivalent to the square of the straight line drawn from the vertex to any point of the base, plus the product of the segments of the base (**211**), (**212**).

4. The square of the hypotenuse of a right triangle is equivalent to four times the triangle, plus the square of the difference of the other sides.

The following are to be solved by algebra:

5. The base of a right triangle is b, the sum of the hypotenuse and perpendicular is s; required the hypotenuse and perpendicular.

6. Given the hypotenuse and the sum of the base and perpendicular of a right triangle, to find the base and perpendicular.

7. Given the medial lines of a triangle, to find the sides.

8. Given the base of a triangle, the medial line to the base, and the sum of the other sides, to find those sides.

9. To determine a right triangle, given the hypotenuse and radius of the inscribed circle.

10. If a, b, c, be the sides of a triangle, p its perimeter, prove that the area is $\sqrt{\frac{1}{2}p(\frac{1}{2}p - a)(\frac{1}{2}p - b)(\frac{1}{2}p - c)}$.

II. PROPORTIONALITY AND SIMILARITY.

216. Definitions.

1. The **internal segments** of a line are the distances from the extremities of the line to a point on the line.

2. The **external segments** of a line are the distances from the extremities of the line to a point on the line produced.

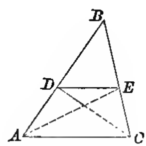
3. **Similar polygons** are mutually equiangular polygons whose corresponding sides are proportional.

4. **Homologous points, lines,** or **angles,** in similar polygons, are points, lines, or angles similarly situated.

5. The **ratio of similitude** of two similar polygons is the ratio of two homologous sides.

217. Proposition XV.—Theorem.

A parallel to one side of a triangle, intersecting the other sides, divides them into proportional internal segments.



Let ABC be a triangle, and DE a parallel to AC. Draw AE and CD.

The triangles, BED, AED, having a common vertex E, and their bases, BD, AD, in the same line BA, have a common altitude — the perpendicular from E to BA — and are therefore proportional to their bases (**200**, 3).

$$\therefore \quad BED : AED :: BD : AD.$$

The triangles, BDE, CDE, having a common vertex D, and their bases, BE, CE, in the same line BC, have a common altitude — the perpendicular from D to BC.

$$\therefore \quad BDE : CDE :: BE : CE.$$

The two triangles, AED, CDE, having a common base DE, and their vertices, A, C, in the line AC, parallel to the base, have equal altitudes — the distances between the parallels at A and C — and are therefore equivalent.

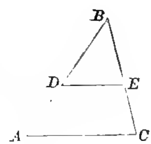
Hence, the two proportions have a ratio of the one equal to a ratio of the other, and therefore the remaining ratios are equal.

$$\therefore \quad BD : AD :: BE : CE.$$

218. Corollaries.

1. *The two sides of a triangle which are divided into internal segments by a line parallel to the third side are proportional to the corresponding segments.*

For, $\quad BD : AD :: BE : CE$ (**217**).



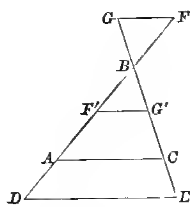
Then, $BD + AD : BD :: BE + CE : BE$,

and $BD + AD : AD :: BE + CE : CE$.

Hence, $\quad BA : BD :: BC : BE$,

and $\quad BA : AD :: BC : CE$.

2. *A parallel to one side of a triangle intersecting the other sides produced divides them into proportional external segments.*



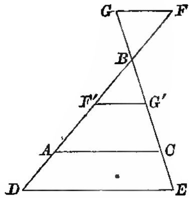
Let ABC be a triangle, and let DE, parallel to AC, intersect the sides, BA, BC, produced. Then,

$$BD : AD :: BE : CE \quad (\textbf{218, } 1).$$

Let FG, parallel to AC, intersect AB and CB produced beyond the vertex.

The alternate angles, BFG, BAC, are equal (**46**, 2).

Revolve FBG, in its own plane, around the vertex B as a center, till it takes the position $F'BG'$ on ABC. Then, since the corresponding angles, $BF'G'$, BAC, are equal, $F'G'$ is parallel to AC (**44,** 4).



$\therefore \quad BA : BF' :: BC : BG'$ (?)

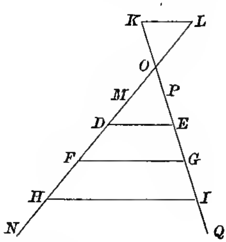
This proportion will be true when the triangle is revolved to its original position, since the revolution will not change the length of the lines. Then, changing BA to AB, and BC to CB,

$$AB : BF :: CB : BG.$$

$$\therefore \quad AB + BF : BF :: CB + BG : BG,$$

or, $$AF : BF :: CG : BG.$$

3. *Two lines intersected by any number of parallels are divided proportionally.*



Let MN and PQ be met by the parallels, DE, FG, HI, KL... Then will MN and PQ be divided proportionally.

Suppose the lines intersect at O.

$OF : DF :: OG : EG$ (**218,** 1).

$OF : FH :: OG : GI$ (**217**).

$FO : FL :: GO : GK$ (**218,** 2), by division.

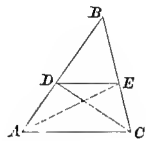
$\therefore \quad DF : FH :: EG : GI,$ (?)

and $FH : FL :: GI : GK.$ (?)

If the lines are parallel, they will not intersect; but then, $DF = EG$, $FH = GI$, $FL = GK$, and the above proportions will be true.

219. Proposition XVI.—Theorem.

A straight line which divides two sides of a triangle into proportional internal segments is parallel to the third side.



Let DE divide the sides, AB, CB, of the triangle ABC, so that

$$BD : AD \quad :: \quad BE : CE.$$

Drawing AE, CD, we have

$$BED : AED :: \quad BD : AD \quad (\mathbf{200},\ 3).$$

Also, $\quad BDE : CDE :: \quad BE : CE.$

But, $\quad BD : AD \quad :: \quad BE : CE.$

$\therefore \quad BED : AED :: BDE : CDE.$ (?)

In this proportion the antecedents are identical; hence, the consequents, AED, CDE, are equivalent; (?) and, since these triangles have a common base DE, their altitudes are equal; that is, their vertices, A, C, points of the line AC, are equally distant from DE, or the prolongations of DE; hence, DE and AC are parallel (**87**).

220. Corollaries.

1. *A straight line which divides two sides of a triangle into internal segments proportional to those sides is parallel to the third side.*

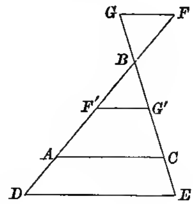
If $\quad BA : AD :: BC : CE.$

Then, $\quad BA - AD : AD :: BC - CE : CE,$

or, $\quad BD : AD :: BE : CE.$

Hence, DE is parallel to AC (**219**).

2. *A straight line which divides two sides of a triangle into proportional external segments is parallel to the third side.*



In the triangle ABC, let DE meet the sides BA and BC produced, so that we have $BD : AD :: BE : CE$. Then, AC and DE are parallel (**220,** 1).

Let FG intersect the sides, AB, CB, produced, so that

$$AF : BF :: CG : BG.$$

Then, $$AF - BF : BF :: CG - BG : BG,$$

or, $$AB : BF :: CB : BG.$$

Revolving FBG to the position $F'BG'$ on ABC, and writing BA for AB, and BC for CB, we have

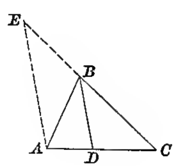
$$BA : BF' :: BC : BG'.$$

Hence, $F'G'$ is parallel to AC; (?) hence, the angle $BF'G'$ is equal to BAC (**46,** 4).

Revolving the triangle to its original position, the angle BFG is equal to BAC, since it has not changed in the revolution; therefore, FG is parallel to AC (**44,** 2).

221. Proposition XVII.—Theorem.

The bisector of any angle of a triangle divides the opposite side into internal segments proportional to the adjacent sides.



Let ABC be a triangle, and BD the bisector of the angle ABC.

Draw AE parallel to DB, meeting CB produced in E.

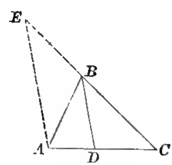
The angles, BAE, ABD, are equal (**46,** 2), also, the angles, BEA, CBD (**46,** 4); but, by hypothesis, the angles, ABD, CBD, are equal; hence, the angles, BAE, BEA, are equal; $\therefore$ $EB = AB$ (**61**).

But, $CD : AD :: CB : EB$ (**217**).

$\therefore$ $CD : AD :: CB : AB$ (**22,** 3).

222. Corollary.

A straight line drawn from the vertex of any angle of a triangle, dividing the opposite side into internal segments proportional to the adjacent sides, is the bisector of that angle.



If $CD : AD :: CB : AB$. Then, BD is the bisector of ABC.

Produce CB till $EB = AB$, and draw AE. Now, substituting EB for its equal AB in the proportion, we have

$$CD : AD :: CB : EB.$$

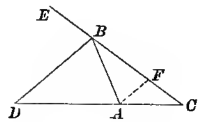
Hence, BD and EA are parallel (**219**); hence, the angles, BAE, ABD, are equal (**46,** 2); also, the angles, BEA, CBD (**46,** 4); and, since $EB = AB$, the angles, BAE, BEA, are equal (**59**); hence, the angles, ABD, CBD, are equal; that is, BD is the bisector of the angle ABC.

223. Proposition XVIII.—Theorem.

The bisector of any exterior angle of a triangle divides the opposite side into external segments proportional to the adjacent sides.

Let ABC be a triangle, BD the bisector of the exterior angle ABE.

Draw AF parallel to DB.



The angles, BAF, ABD, are equal (**46**, 2), also, the angles, BFA, EBD (**46**, 4); but, by hypothesis, the angles, ABD, EBD, are equal; hence, the angles, BAF, BFA, are equal; hence, $FB = AB$ (**61**).

But, $\quad CD : AD :: CB : FB \quad$ (**218**, 1).

$\therefore \quad CD : AD :: CB : AB \quad$ (**22**, 3).

224. Corollary.

A straight line drawn from the vertex of any angle of a triangle, dividing the opposite side into external segments proportional to the adjacent sides, is the bisector of the exterior angle. (?)

225. Scholiums.

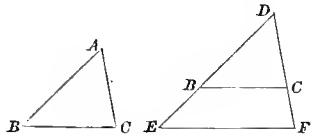
1. In case the triangle ABC is isosceles, AB and BC being the equal sides, the angles, BAC, BCA, are equal; and since $ABE = BAC + BCA$, and $EBD = \frac{1}{2}ABE$, $EBD = BCA$; and hence the bisector BD is parallel to CA (**44**, 4), and will not intersect it, and the segments, CD, AD, become infinite.

2. If $BA > BC$, the angle $BCA >$ the angle BAC; $\therefore$ $DBE < BCA$; and DB produced will bisect the exterior angle formed by producing AB, and intersect AC produced, dividing it externally as in (**223**).

226. Proposition XIX.—Theorem.

Mutually equiangular triangles are similar.

In the triangles, ABC, DEF, let $A = D$, $B = E$, and $C = F$. Place ABC on DEF so that the angles, A, D, shall coincide, and the triangle ABC take the position DBC.



Since the angle $DBC = E$, BC is parallel to EF. (?)

Hence, $DB : DE :: DC : DF$ (**218**, 1).

Substituting AB for DB and AC for DC, we have

$$AB : DE :: AC : DF.$$

In like manner, by placing ABC on DEF so that the equal angles, B, E, shall coincide, we shall have

$$AB : DE :: BC : EF.$$

These proportions have a common ratio; hence, the other ratios are equal, and we have the continued proportion,

$$AB : DE :: AC : DF :: BC : EF.$$

$\therefore$ the triangles, ABC, DEF, are similar (**216**, 3).

227. Corollaries.

1. *Two triangles are similar when two angles of the one are respectively equal to two angles of the other.* (?)

2. *In similar triangles, the homologous sides lie opposite equal angles.*

3. *The ratio of similitude of two similar triangles is the ratio of any two homologous lines.* (?)

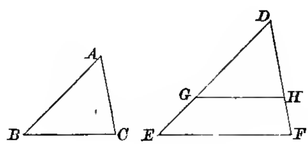
228. Proposition XX.—Theorem.

Triangles whose corresponding sides are proportional are similar.

In the triangles, ABC, DEF, let

$$AB : DE :: AC : DF :: BC : EF.$$

Take $DG = AB$, and draw GH parallel to EF. Then, the triangles, DGH, DEF, are mutually equiangular, and, therefore, similar (**226**).



$$\therefore \quad DG : DE :: DH : DF :: GH : EF \quad (\mathbf{216},\ 3).$$

Since $AB = DG$, the first ratios of the two continued proportions are equal; hence, the other ratios are equal.

$$\therefore \quad \begin{cases} AC : DF :: DH : DF, \ \therefore \ AC = DH. \\ BC : EF :: GH : EF, \ \therefore \ BC = GH. \end{cases}$$

Hence, the triangles, ABC, DGH, are mutually equilateral, and therefore equal (**75**); but DGH and DEF are similar; therefore, ABC and DEF are similar.

229. Scholium.

In order to establish the similarity of two polygons, it must be proved that they fulfill the two conditions (**216**, 3):

1st. *They must be mutually equiangular.*

2d. *Their corresponding sides must be proportional.*

In the case of triangles, either of these conditions involves the other; hence, to establish the similarity of two triangles, it will suffice to prove either that they are mutually equiangular, or that their corresponding sides are proportional.

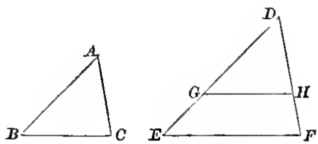
But to establish the similarity of polygons having more than three sides, both conditions must be proved to be fulfilled, since, in this case, equality of angles does not involve proportionality of sides; nor does proportionality of sides involve equality of angles.

230. Proposition XXI.—Theorem.

Two triangles having an angle of the one equal to an angle of the other, and the including sides proportional, are similar.

In the triangles, ABC, DEF, let $A = D$, and

$$AB : DE :: AC : DF.$$



Then, these triangles are similar.

Take $DG = AB$, $DH = AC$, and draw GH. Then, the triangles, ABC, DGH, are equal (**55**).

Substituting DG for AB, and DH for AC, in the proportion above,

$$DG : DE :: DH : DF.$$

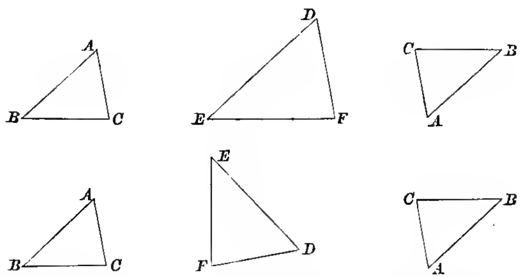
Hence, GH is parallel to EF (**220,** 1); hence, the triangles, DGH, DEF, are equiangular (**46,** 4), and, therefore, similar (**226**).

But the triangles, ABC, DGH, are equal; hence, ABC and DEF are similar.

231. Proposition XXII.—Theorem.

Triangles having their sides respectively parallel or perpendicular are similar.

Let ABC and DEF be the triangles; A and D, B and E, C and F, the pairs of angles whose sides are respectively parallel or perpendicular.



The angles of these pairs are equal or supplemental (**48**), (**49**). Then, denoting a right angle by R, we shall have one of the following cases:

1. $A + D = 2R, \quad B + E = 2R, \quad C + F = 2R.$

2. $A = D, \quad B + E = 2R, \quad C + F = 2R.$

3. $A = D, \quad B = E, \quad \therefore \quad C = F.$

If the first case is true, the sum of all the angles of the two triangles would be six right angles, which is impossible, since this sum is four right angles (**52**); hence, the first case is false.

If the second case is true, the sum of all the angles of the two triangles exceeds four right angles by $A + D$, which is impossible; hence, the second case is false.

Therefore, the third case is true; that is, the triangles are equiangular, and hence, similar (**226**).

232. Scholiums.

1. In two triangles whose sides are respectively parallel, the homologous sides are those which are parallel.

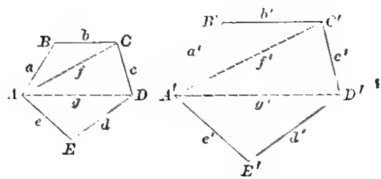
2. In two triangles whose sides are respectively perpendicular, the homologous sides are those which are perpendicular.

3. In two triangles whose sides are respectively parallel or perpendicular, the homologous or equal angles are those which are opposite homologous sides.

233. Proposition XXIII.—Theorem.

Polygons composed of the same number of triangles, respectively similar, and similarly placed, are similar.

Let $ABCDE$, $A'B'C'D'E'$, be polygons composed of the same number of triangles respectively similar and similarly placed.



1. The polygons are mutually equiangular.

For, any two corresponding angles of the polygons are either homologous angles of two similar triangles, or sums of two or more such angles. Thus, $B = B'$,

$$BCD = BCA + ACD = B'C'A' + A'C'D' = B'C'D'. \ldots$$

2. The corresponding sides of the polygons are proportional.

For, denoting the corresponding sides and diagonals by like letters, as in the diagrams, accenting those of the

second, we have, from the similar triangles, the continued proportion,

$$\frac{a}{a'} = \frac{b}{b'} = \frac{f}{f'} = \frac{c}{c'} = \frac{g}{g'} = \frac{d}{d'} = \frac{e}{e'}.$$

Omitting the ratios of the diagonals, we may write

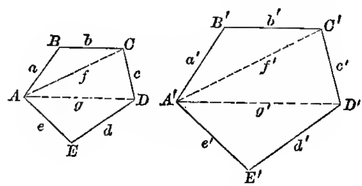
$$a : a' :: b : b' :: c : c' :: d : d' :: e : e'.$$

Hence, the polygons are similar (**216**, 3).

234. Proposition XXIV.—Theorem.

Similar polygons can be decomposed into the same number of triangles, respectively similar and similarly placed.

Let $ABCDE$, $A'B'C'D'E'$, be two similar polygons, whose homologous sides, angles, and diagonals are denoted by like letters, those of the second polygon being accented.



From two homologous vertices, as A, A', let diagonals be drawn to the vertices of the opposite angles.

Since the polygons are similar, they are mutually equiangular, and have their homologous sides proportional.

$$\therefore \quad B = B', \text{ and } a : a' :: b : b'.$$

Hence, the triangles, ABC, $A'B'C'$, are similar (**230**).

These triangles are also similarly placed in the polygons. The angle BCA, opposite the side a, is equal to the angle $B'C'A'$, opposite the homologous side a'.

Since the angle $BCD = B'C'D'$, and $BCA = B'C'A'$,

$$BCD - BCA = B'C'D' - B'C'A', \quad \therefore \quad ACD = A'C'D'.$$

From the similar triangles and polygons, we have

$$b : b' :: f : f', \text{ and } b : b' :: e : e'.$$

$$\therefore \quad f : f' \quad :: \quad e : e'.$$

Hence, the triangles, ACD, $A'C'D'$, are similar. (?) These triangles are also similarly placed in the polygons.

In like manner, the remaining triangles of the one polygon can be proved to be similar to those similarly placed in the other.

235. Scholiums.

1. Similar polygons can be decomposed into the same number of triangles, respectively similar and similarly placed, by drawing lines from any two homologous points to the vertices, or to the vertices and other homologous points in the perimeters.

2. If the points from which the lines are drawn are without the perimeters, the polygons will be respectively the sums of the triangles partly interior and partly exterior, minus the sums of those wholly exterior.

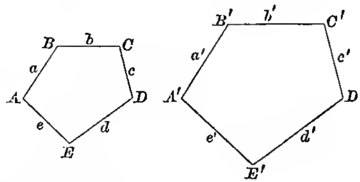
3. The ratio of any two homologous lines is equal to the ratio of any two homologous sides, and is, therefore, equal to the ratio of similitude of the polygons.

4. Two similar polygons are equal when any two homologous lines are equal.

236. Proposition XXV.—Theorem.

The perimeters of similar polygons are proportional to any two homologous lines.

Let $ABCDE$, $A'B'C'D'E'$, be two similar polygons whose homologous sides are denoted by like letters, those of the second being accented, and whose perimeters are denoted by p, p', respectively.



For, since the polygons are similar,

$$a : a' :: b : b' :: c : c' :: d : d' :: e : e'.$$

$$\therefore \quad a + b + c + d + e : a' + b' + c' + d' + e' :: a : a',$$

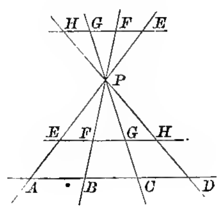
$$\text{or,} \qquad p : p' :: a : a' :: b : b' \ldots$$

Hence, the perimeters are proportional to any two homologous lines (**235,** 3).

237. Proposition XXVI.—Theorem.

If three or more straight lines passing through a common point intersect two parallels, the corresponding segments of the parallels are proportional.

Let AE, BF, CG, DH, passing through the point P, intersect the parallels, AD, EH, on the same side or on opposite sides of P.



The triangles, APB, EPF, are similar, also, BPC, FPG, and CPD, GPH. (?)

$$\therefore \quad \left\{ \begin{array}{l} PB : PF :: AB : EF \\ PB : PF :: BC : FG \end{array} \right\} \therefore AB : EF :: BC : FG,$$

$$\text{and} \quad \left\{ \begin{array}{l} PC : PG :: BC : FG \\ PC : PG :: CD : GH \end{array} \right\} \therefore BC : FG :: CD : GH.$$

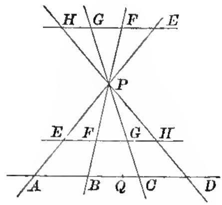
$$\therefore \quad AB : EF :: BC : FG :: CD : GH.$$

238. Proposition XXVII.—Theorem.

Three or more non-parallel straight lines dividing two parallels proportionally pass through a common point.

Let the non-parallel straight lines, AE, BF, CG, DH, divide the parallels, AD, EH, so that

$$AB : EF :: BC : FG :: CD : GH.$$



Since these lines are not parallel, any two, as AE, BF, intersect at some point P.

Now, it is to be proved that CG and DH pass through the same point.

Draw PG; then, PG produced will pass through C; for, if not, let PG produced intersect AD in some other point, as Q. By hypothesis,

$$AB : EF :: BC : FG, \quad \therefore \quad BC = \frac{AB \times FG}{EF}.$$

If PG produced passes through Q, we have by (**237**),

$$AB : EF :: BQ : FG, \quad \therefore \quad BQ = \frac{AB \times FG}{EF}.$$

$\therefore$ $BQ = BC$, which is impossible; hence, PG passes through C; that is, CG passes through P.

In like manner, it can be proved that DH passes through P.

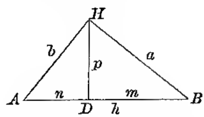
239. Proposition XXVIII.—Theorem.

If from the vertex of the right angle of a right triangle a perpendicular be drawn to the hypotenuse, then,

1. *The two triangles into which the given triangle is thus divided are similar to the given triangle and to each other.*

2. *The perpendicular is a mean proportional between the segments into which it divides the hypotenuse.*

3. *Each side adjacent to the right angle is a mean proportional between the hypotenuse and the adjacent segment.*



Let AHB be a triangle right angled at H; a, h, b, the sides respectively opposite the angles, A, H, B; p, the perpendicular from the vertex of the right angle to the hypotenuse; m and n, segments of the hypotenuse respectively adjacent to a and b.

1. The triangle BDH is similar to the triangle AHB; for, the angle B is common, the right angle BDH is equal to the right angle AHB, and, therefore, the angle BHD is equal to A (**53**, 9), (**226**).

In like manner, it can be proved that the triangle ADH is similar to the triangle AHB.

Therefore, the triangles, BDH, ADH, are similar.

2. In the similar triangles, BDH, ADH, m and p are homologous, since they are respectively opposite the equal angles, BHD and A; p and n are homologous, since they are respectively opposite the equal angles, B and AHD.

$$\therefore \quad m : p :: p : n, \quad \therefore \quad p^2 = mn.$$

3. In the similar triangles, AHB, BDH, h and a are homologous, since they are respectively opposite the equal angles, AHB and BDH; a and m are homologous, since they are respectively opposite the equal angles, A and BHD.

$$\therefore \quad h : a :: a : m, \quad \therefore \quad a^2 = hm.$$

In like manner, it can be shown that

$$h : b :: b : n, \quad \therefore \quad b^2 = hn.$$

240. Corollaries.

1. *The squares of the sides adjacent to the right angle are proportional to the adjacent segments of the hypotenuse.*

For, $\frac{a^2}{b^2} = \frac{hm}{hn} = \frac{m}{n}$, $\therefore$ $a^2 : b^2 :: m : n$.

2. *The sum of the squares of the sides of a right triangle adjacent to the right angle is equivalent to the square of the hypotenuse.*

For, $a^2 + b^2 = hm + hn = h(m + n) = h \times h = h^2$.

3. *If from any point in the circumference of a circle a perpendicular be drawn to a diameter, and chords from the same point to the extremities of the diameter, then,*

1st. *The perpendicular is a mean proportional between the segments of the diameter.*

2d. *Each chord is a mean proportional between the diameter and the segment adjacent to the chord.*

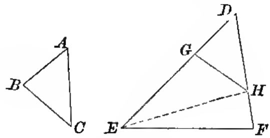
For, the triangle formed by the diameter and chords is right angled (**154,** 2); hence, the statements of the corollary are identical with those of **239,** 2, 3.

3d. *The sum of the squares of the chords is equivalent to the square of the diameter.* (?)

241. Proposition XXIX.—Theorem.

Triangles having an angle of the one equal to an angle of the other are proportional to the products of the sides including the equal angles.

Let ABC and DEF be two triangles having the angle A equal to the angle D. Take $DG = AB$, $DH = AC$, and draw GH; then, the triangles, ABC and DGH, are equal (**55**).



Drawing EH, the two triangles, DGH, DEH, having the common vertex H, and their bases, DG, DE, in the same line, have a common altitude — the perpendicular from H to DE.

$$\therefore \quad DGH : DEH :: DG : DE \quad (\mathbf{200},\ 3).$$

In like manner, it can be proved that

$$DEH : DEF :: DH : DF.$$

Taking the product of the corresponding terms of these proportions, omitting DEH, the common factor of the first couplet, we have

$$DGH : DEF :: DG \times DH : DE \times DF.$$

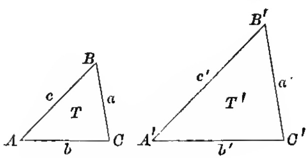
Substituting ABC for DGH, and $AB \times AC$ for $DG \times DH$, we have

$$ABC : DEF :: AB \times AC : DE \times DF.$$

242. Proposition XXX.—Theorem.

Similar triangles are proportional to the squares of any two homologous sides.

Let ABC, $A'B'C'$, be similar triangles; A and A', B and B', C and C', pairs of equal angles; a and a', b and b', c and c', pairs of homologous sides; T and T', respectively, the areas of the triangles.



Since the triangles are mutually equiangular, we have, by (**241**),

$$T : T' :: ab : a'b' :: bc : b'c' :: ca : c'a'.$$

Therefore,

$$\frac{T}{T'} = \frac{ab}{a'b'} = \frac{bc}{b'c'} = \frac{ca}{c'a'} = \frac{a}{a'} \times \frac{b}{b'} = \frac{b}{b'} \times \frac{c}{c'} = \frac{c}{c'} \times \frac{a}{a'}.$$

Since the triangles are similar, we have, by (**216,** 3),

$$a : a' :: b : b' :: c : c', \therefore \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

Substituting $\frac{a}{a'}$ for $\frac{b}{b'}$, $\frac{b}{b'}$ for $\frac{c}{c'}$, $\frac{c}{c'}$ for $\frac{a}{a'}$, we have

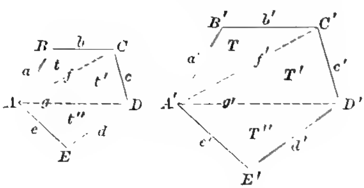
$$\frac{T}{T'} = \frac{a^2}{a'^2} = \frac{b^2}{b'^2} = \frac{c^2}{c'^2}.$$

$$\therefore \quad T : T' :: a^2 : a'^2 :: b^2 : b'^2 :: c^2 : c'^2.$$

243. Proposition XXXI.—Theorem.

Similar polygons are proportional to the squares of any two homologous sides.

Let $ABCDE$, $A'B'C'D'E'$, be two similar polygons; A and A', B and B' . . . , pairs of equal angles; a and a', b and b' . . . , pairs of homologous sides; f and f', g and g', pairs of homologous diagonals; t and T, t' and T' . . . , pairs of similar triangles (**234**); P and P', the areas of the polygons. Then,



$$P : P' :: a^2 : a'^2 :: b^2 : b'^2 \ldots$$

For, $\left\{\begin{array}{l} t : T :: f^2 : f'^2 \\ t' : T' :: f^2 : f'^2 \end{array}\right\} \therefore \quad t : T :: t' : T'.$

Also, $\left\{\begin{array}{l} t' : T' :: g^2 : g'^2 \\ t'' : T'' :: g^2 : g'^2 \end{array}\right\} \therefore \quad t' : T' :: t'' : T''.$

$\therefore \qquad t : T :: t' : T' :: t'' : T''.$

$\therefore \qquad t + t' + t'' : T + T' + T'' :: t : T :: a^2 : a'^2.$

But, $\quad t + t' + t'' = P,$ and $T + T' + T'' = P',$

and $\quad a^2 : a'^2 :: b^2 : b'^2 :: c^2 : c'^2 \ldots.$ (?)

$\therefore \qquad P : P' :: a^2 : a'^2 :: b^2 : b'^2 :: c^2 : c'^2 \ldots.$

244. Corollaries.

1. *Similar polygons are proportional to the squares of any two homologous lines.* (?)

2. *If similar polygons be constructed on the three sides of a right triangle as homologous sides, the polygon on the hypotenuse is equivalent to the sum of the polygons on the other sides.*

Let P, P', P'', be the similar polygons constructed on the hypotenuse h, and the sides a and b respectively.

Then, $P : P'' :: h^2 : b^2$ (**243**).

Also, $P' : P'' :: a^2 : b^2, \quad \therefore \quad P' + P'' : P'' :: a^2 + b^2 : b^2.$

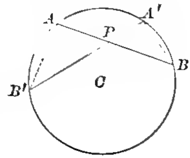
But, $a^2 + b^2 = h^2$ (**240**, 2), $\quad \therefore \quad P' + P'' : P'' :: h^2 : b^2.$

$\therefore \qquad P : P'' :: P' + P'' : P'',$ (?) $\quad \therefore \quad P = P' + P''.$ (?)

245. Proposition XXXII.—Theorem.

If through a fixed point within a circle two chords be drawn, the rectangle of the segments of one chord is equivalent to the rectangle of the segments of the other.

Let AB, $A'B'$, be two chords drawn through any fixed point P, within the circle C. Then,



$$PA \times PB = PA' \times PB'.$$

Drawing the chords, AB', $A'B$, the angles, APB', $A'PB$, are equal (**33**); the angles, A, A', are equal (**154**, 1); the angles, B', B, are equal; (?) hence, the triangles, APB', $A'PB$, are mutually equiangular, and therefore similar.

The sides, PA, PA', respectively opposite the equal angles, B', B, are homologous; also, the sides, PB', PB, respectively opposite the equal angles, A, A'.

$$\therefore \quad PA : PA' :: PB' : PB,$$

$$\therefore \quad PA \times PB = PA' \times PB'.$$

246. Corollary.

If a chord passing through a fixed point within a circle revolve about this point as a center, the area of the rectangle of the varying segments is constant and equivalent to the square of one-half of the least chord passing through the same point. (?)

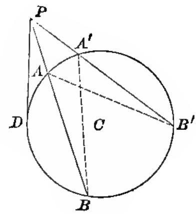
247. Proposition XXXIII.—Theorem.

If from a fixed point without a circle any two secants be drawn terminating in the circumference, the rectangle of one

secant and its external segment is equivalent to the rectangle of the other secant and its external segment.

Let PB, PB', be two secants drawn from the point P without the circle C, terminating in its circumference. Then,

$$PB \times PA = PB' \times PA'.$$



Drawing the chords, $A'B$, AB', the two triangles, $A'PB$, APB', having the angle P common, the angles, B', B, equal, are similar (**227**, 1).

$$\therefore \quad PB : PB' :: PA' : PA;$$

$$\therefore \quad PB \times PA = PB' \times PA'.$$

248. Corollary.

If a secant drawn from a fixed point without a circle, terminating in the circumference, revolve about this point as a center, the area of the rectangle of the varying secant and its external segment is constant, and equivalent to the square of the tangent drawn from the same point to the circumference; that is, the tangent is a mean proportional between the secant and its external segment. (?)

249. Exercises.

1. If two circles are tangent internally, chords of the greater, drawn from the point of tangency, are divided proportionally by the circumference of the less.

2. If two circles are tangent externally, secants drawn through their point of tangency, terminating in the circumferences, are divided proportionally at the point of tangency.

3. If two circles are tangent externally, the part of their common tangent included between their points of contact is a mean proportional between the diameters.

4. The prolongation of the common chord of two intersecting circles bisects their common tangent.

5. If two circles intersect, the greatest line which can be drawn through either point of intersection, terminating in the circumferences, is parallel to the line joining their centers.

6. The rectangle of two sides of any triangle is equivalent to the rectangle of the diameter of the circumscribed circle and the perpendicular to the third side from the vertex of the opposite angle.

7. The rectangle of any two sides of a triangle is equivalent to the rectangle of the internal segments of the third side formed by the bisector of the opposite angle, plus the square of the bisector.

8. The rectangle of any two sides of a triangle is equivalent to the rectangle of the external segments of the third side formed by the bisector of the opposite exterior angle, minus the square of the bisector.

9. The area of a triangle is equal to the product of its three sides divided by twice the diameter of the circumscribed circle.

10. The rectangle of the diagonals of an inscribed quadrilateral is equivalent to the sum of the rectangles of the opposite sides.

11. If a fixed circumference is cut by any circumference passing through two fixed points, the common chord passes through a fixed point (**248**).

12. Through a given point within a circle draw a chord so that the point shall divide the chord into segments having a given ratio (**249,** 1).

13. From a point without a circle draw a secant terminating in the circumference, such that the secant shall have a given ratio to its external segment (**249,** 2).

14. Describe a circumference which shall pass through two given points and be tangent to a given straight line (**248**).

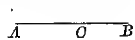
15. Describe a circumference which shall be tangent to a given circumference, have its center in a given straight line, and pass through a given point of that line (**70,** 2).

III. CONSTRUCTIONS.

250. Definitions.

1. A line is divided *internally in extreme and mean ratio*, when the greater internal segment is a mean proportional between the given line and the less segment.

Thus, AB is divided internally at C in extreme and mean ratio, if

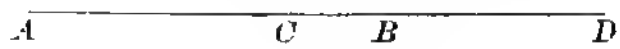


$$AB : AC :: AC : BC.$$

2. A line is divided *externally in extreme and mean ratio*, when the less external segment is a mean proportional between the given line and the greater segment. Thus, AB is divided externally at D in extreme and mean ratio, if

$$AB : BD :: BD : AD.$$

3. A line is divided *harmonically*, when it is divided internally and externally so that the ratio of the internal segments is equal to the ratio of the external segments. Thus, AB is divided harmonically at C and D, if



$$AC : BC :: AD : BD.$$

This proportion taken by alternation gives

$$AC : AD :: BC : BD.$$

This proportion affords a second definition of a line divided harmonically, as follows:

4. A line is divided harmonically, when it is divided internally and externally so that the ratio of the distances from one extremity to the points of division is equal to the ratio of the distances from the other extremity to the same points.

The last proportion also proves that the line CD is divided harmonically at the points B and A (**250**, 3).

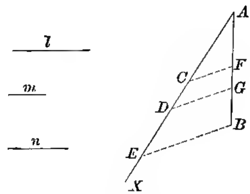
5. **Harmonic points** are the extremities and division points of a line divided harmonically. Thus, A, B, C, D, on the line above, are harmonic points.

6. **Conjugate points** are the extremities and division points of lines divided harmonically, taken in pairs. Thus, A and B are conjugate points; also, C and D.

7. The **center of similitude** of similar polygons is a point such that lines drawn from it to the vertices of one of the polygons will respectively pass through the homologous vertices of the other polygon.

251. Proposition XXXIV.—Problem.

To divide a given straight line into parts proportional to given straight lines.



Let it be required to divide AB into parts proportional to l, m, n. From A draw the indefinite line AX, making a convenient angle with AB.

On AX lay off $AC = l$, $CD = m$, $DE = n$. Draw EB and CF, DG parallel to EB. Then, AF, FG, GB, are proportional to l, m, n (**218**, 3).

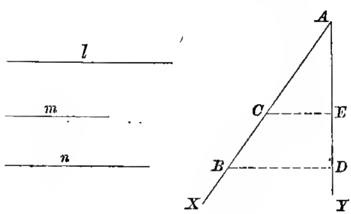
252. Corollary.

To divide a given straight line into any number of equal parts.

Make the lines, l, m, n . . . , of the last article equal.

253. Proposition XXXV.—Problem.

To find a fourth proportional to three given straight lines.



Let it be required to find a fourth proportional to l, m, n. Draw the indefinite straight lines, AX, AY, making a convenient angle with each other.

On AX lay off $AB = l$, $AC = m$, and on AY lay off $AD = n$. Draw BD, and

CE parallel to BD. Then, AE is the fourth proportional required.

For, $AB : AC :: AD : AE$; $\therefore\ l : m :: n : AE$.

254. Corollary.

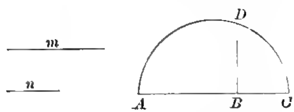
To find a third proportional to two given straight lines.

Let it be required to find a third proportional to l and m. Find a fourth proportional to l, m, and m.

255. Proposition XXXVI.—Problem.

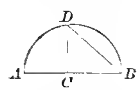
To find a mean proportional between two given straight lines.

Let it be required to find a mean proportional between m and n.



1*st method.* Make $AB = m$, and produce it till $BC = n$. On AC as a diameter describe a semi-circumference. At B draw BD perpendicular to AC. Then, BD is the required mean proportional (**240,** 3, 1st).

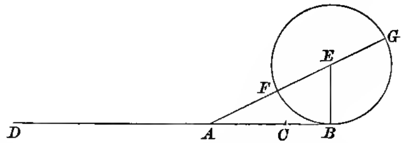
2d method. Make $AB = m$, the greater line. From B lay off $BC = n$. At C draw CD perpendicular to AB, and extend it to the semi-circumference on AB, and draw BD. Then, BD is the required mean proportional (**240,** 3, 2d).



256. Proposition XXXVII.—Problem.

To divide a given straight line internally and externally in extreme and mean ratio.

Let AB be the given line. At B draw to AB the perpendicular BE equal to one-half of AB. With E as center and EB as radius, describe a circumference. Draw the straight line AEG, cutting the circumference in F and G.



On AB lay off $AC = AF$, and on the prolongation of BA lay off $AD = AG$. Then, the line AB is divided internally at C and externally at D, in extreme and mean ratio.

For, since AB is a tangent (**133**), and AG a secant, AB is a mean proportional between AG and AF (**248**).

$\therefore \quad AG : AB :: AB : AF.$

$\therefore \quad AG - AB : AB :: AB - AF : AF.$

Now, $\quad FG = AB$ (?), also, $AC = AF.$

$\therefore \quad AC : AB :: BC : AC.$

$\therefore \quad AB : AC :: AC : BC.$

The first proportion by composition gives

$$AG + AB : AG :: AB + AF : AB.$$

Since $AG = AD$, and $AB = FG$, we have

$$BD : AD :: AD : AB.$$

257. Corollary.

To find the numerical value of the segments.

Let l denote the line AB.

(1) $AC = AE - EF.$ (2) $BC = BA - AC.$

(3) $AD = AE + EG.$ (4) $BD = BA + AD.$

(5) $AE = \sqrt{l^2 + \frac{1}{4}l^2} = \frac{1}{2}l\sqrt{5}.$ (6) $EF = EG = \frac{1}{2}l.$

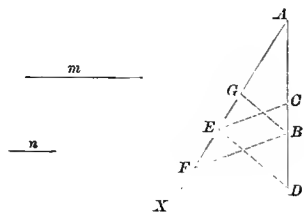
Substituting (5) and (6) in (1) and (3), and the results in (2) and (4), we have

(7) $AC = \frac{1}{2} l (\sqrt{5} - 1)$. (8) $BC = \frac{1}{2} l (3 - \sqrt{5})$.

(9) $AD = \frac{1}{2} l (\sqrt{5} + 1)$. (10) $BD = \frac{1}{2} l (3 + \sqrt{5})$.

258. Proposition XXXVIII.—Problem.

To divide a given straight line harmonically in a given ratio.



Let it be required to divide AB harmonically in the ratio of m to n. Draw AX, making a convenient angle with AB.

On AX lay off $AE = m$, and from E lay off EF and EG each equal to n. Draw FB and GB, and EC parallel to FB, and ED parallel to GB. Then, C and D are the points of division required.

For, $\quad AC : BC :: AE : FE :: m : n$.

Also, $\quad AD : BD :: AE : GE :: m : n$.

259. Corollaries.

1. *Given the harmonic ratio and the extreme harmonic points, to find the conjugate of each of the extreme points.*

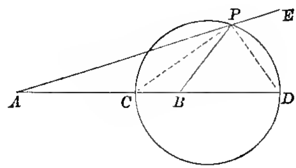
Take, as in the figure of the last article, $AE = m$, $EF = n$, $EG = n$. Draw ED; then, GB, parallel to ED, will give B, the conjugate of A. Draw FB; then, EC, parallel to FB, will give C, the conjugate of D.

2. *If any three of four harmonic points be given, the fourth can be found from the proportion,*

$$AC : BC :: AD : BD.$$

260. Proposition XXXIX.—Problem.

To find the locus of all the points whose distances from two given points are in a given ratio.



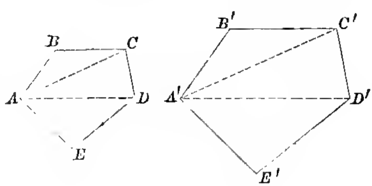
Let A and B be the given points, and $m : n$ the given ratio. Divide AB harmonically in the given ratio at the points, C, D (**258**). Let P be a point of the required locus. Then,

$$AP : BP :: AC : BC :: AD : BD :: m : n.$$

Hence, PC is the bisector of the angle APB (**222**), and PD is the bisector of the exterior angle BPE (**224**). But the bisectors, PC, PD, are perpendicular to each other (**34**, 3, 4). Hence, P is in the circumference whose diameter is CD (**179**, 3).

261. Proposition XL.—Problem.

On a given line to construct a polygon similar to a given polygon.



Let it be required to construct on the line $A'B'$ a polygon similar to the polygon $ABCDE$.

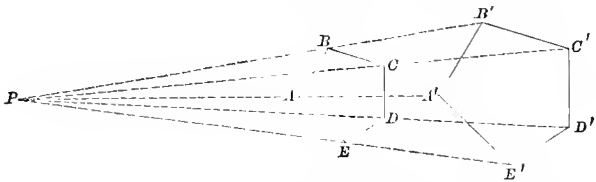
From A draw the diagonals, AC, AD. Construct the angle $A'B'C' = B$, and $B'A'C' = BAC$. Then, the triangle $A'B'C'$ is similar to the triangle ABC (**227**, 1).

In like manner, construct the triangle $A'C'D'$ similar to ACD, and the triangle $A'D'E'$ similar to ADE. Then, the polygon $A'B'C'D'E'$ is similar to the polygon $ABCDE$ (**233**).

262. Proposition XLI.—Problem.

Given the ratio of similitude of two similar polygons, and one of the polygons, to construct the other.

Let $m : n$ be the ratio of similitude, and $ABCDE$ the given polygon. Take any point P for the center of similitude. Draw from P lines through the vertices, A, B, C...



On PA produced lay off PA' equal to the fourth proportional to m, n, and PA (**253**).

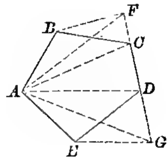
From A' draw $A'B'$ parallel to AB, till it meets PB produced, and from B' draw $B'C'$ parallel to BC, till it meets PC produced, and so on. Then, $A'B'C'D'E'$ is similar to $ABCDE$, and the ratio of similitude is $m : n$.

For, the polygons are equiangular, since their sides are respectively parallel; and the corresponding sides are proportional, since the ratio of any two homologous sides is $m : n$, the ratio of similitude. (?)

263. Proposition XLII.—Problem.

To construct a triangle equivalent to a given polygon.

Let $ABCDE$ be the given polygon. Take any three consecutive vertices, as A, B, C, and draw the diagonal AC. From B draw BF parallel to AC till it meets the prolongation of DC in F, and draw AF.



The triangles, AFC, ABC, are equivalent, for they have a common base AC, and equal altitudes, since their vertices, F, B, lie in the line BF, parallel to AC.

To each of these equivalent triangles add the quadrilateral, $ACDE$, and we shall have the quadrilateral $AFDE$ equivalent to the pentagon $ABCDE$.

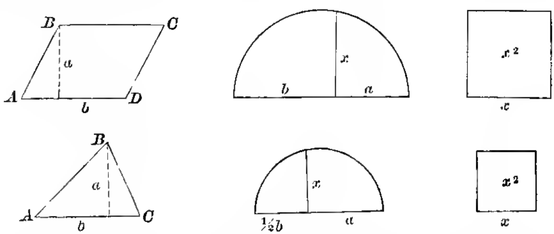
In like manner, we find the triangle AGD equivalent to the triangle AED.

To each of these equivalent triangles add the triangle AFD, and we shall have the triangle AFG equivalent to the quadrilateral $AFDE$, and hence, to the pentagon $ABCDE$.

In like manner, we may find a triangle equivalent to any given polygon, whatever be the number of sides.

264. Proposition XLIII.—Problem.

To construct a square equivalent to a given parallelogram or to a given triangle.



1. Let AC be the parallelogram, a its altitude, b its base. Find a mean proportional x between a and b (**255**).

Then, $a : x :: x : b$; $\therefore\ x^2 = ab$ (**197**).

2. Let ABC be the triangle, a its altitude, b its base. Find a mean proportional x between a and $\frac{1}{2}b$.

Then, $a : x :: x : \frac{1}{2}b$; $\therefore\ x^2 = \frac{1}{2}ab$ (**199**).

265. Corollary.

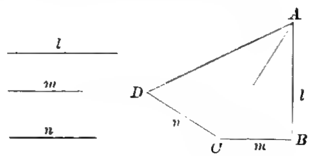
To construct a square equivalent to any given polygon.

Construct a triangle equivalent to the given polygon (**263**), then, a square equivalent to the given triangle (**264**, 2).

266. Proposition XLIV.—Problem.

To construct a square equivalent to the sum of two or more given squares, or to the difference of two given squares.

1. Let l, m, n, be the sides of the given squares. Take $AB = l$, and perpendicular to AB draw $BC = m$, and draw AC.

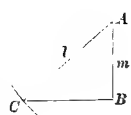


Then, $$\overline{AC}^2 = l^2 + m^2 \quad (\mathbf{209}).$$

Perpendicular to AC, draw $CD = n$, also draw AD.

Then, $\overline{AD}^2 = \overline{AC}^2 + n^2 = l^2 + m^2 + n^2$.

2. Let l and m be the sides of two given squares. Take $AB = m$, and draw BC perpendicular to AB. With A as center and l as radius, describe an arc cutting BC in C.



Then, $$\overline{BC}^2 = l^2 - m^2 \quad (\mathbf{210},\ 1).$$

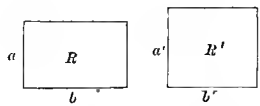
267. Corollary.

To construct a square equivalent to the sum of any number of given polygons, or to the difference of two given polygons.

Construct squares respectively equivalent to the given polygons (**265**), then, a square equivalent to the sum or difference of these squares (**266**).

268. Proposition XLV.—Problem.

On a given line to construct a rectangle equivalent to a given rectangle.



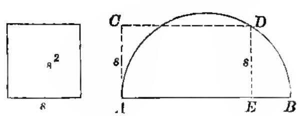
Let R be the given rectangle, a its altitude, b its base, and b' the given line. Find a fourth proportional, a', to b', b, and a (**253**). Then, a' is the altitude of the required rectangle R'.

For, $\quad b' : b :: a : a', \quad \therefore \quad a'b' = ab.$

But, $\quad R' = a'b', \quad R = ab, \quad \therefore \quad R' = R.$

269. Proposition XLVI.—Problem.

Given the area of a rectangle and the sum of two adjacent sides, to construct the rectangle.



Let s^2 be the given area, AB the sum of two adjacent sides of the required rectangle.

On AB as a diameter construct a semi-circumference. At A erect a perpendicular to AB equal to s, and at its extremity draw CD parallel to AB. At D, one point of the intersection of this parallel with the circumference, let

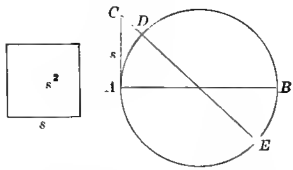
fall the perpendicular DE on AB. Then, AE and EB are the sides of the rectangle.

For, $\overline{AE} \times \overline{EB} = s^2$ (**240**, 3, 1st).

270. Proposition XLVII.—Problem.

Given the area of a rectangle and the difference of two adjacent sides, to construct the rectangle.

Let s^2 be the given area, AB the difference of two adjacent sides of the required rectangle.



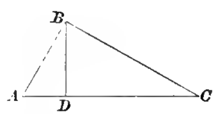
On AB as a diameter construct a circumference. At A erect a perpendicular AC to AB, equal to s, and from its extremity C draw the secant CE through the center. Then, CE and CD are the sides of the rectangle.

For, $CE \times CD = s^2$ (**248**).

271. Proposition XLVIII.—Problem.

To find two straight lines having the same ratio as that of the areas of two given polygons.

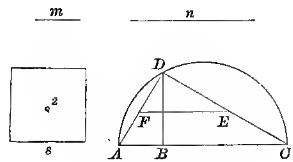
Construct squares equivalent to the given polygons (**265**). Construct a right angle ABC. From B lay off BA and BC, respectively equal to the sides of the squares. Let fall the perpendicular BD. Then, AD, DC, are the required lines.



For, $\overline{AB}^2 : \overline{BC}^2 :: AD : DC$ (**210**, 3).

272. Proposition XLIX.—Problem.

To find a square which shall be to a given square in the ratio of two given straight lines.



Let s^2 be the given square, and $m : n$ the given ratio. On an indefinite straight line lay off $AB = m$ and $BC = n$. On AC as a diameter describe a semi-circumference. At B erect BD perpendicular to AC, cutting the circumference in D, and draw DA and DC. On DC lay off $DE = s$, and draw EF parallel to CA. Then, DF is the side of the required square.

For, $DF : DE :: DA : DC$ (**218**, 1).

$\therefore \overline{DF}^2 : \overline{DE}^2 :: \overline{DA}^2 : \overline{DC}^2$. (?)

Now, $\overline{DA}^2 : \overline{DC}^2 :: AB : BC$ (**210**, 3).

But, $AB = m$ and $BC = n$.

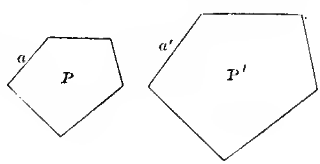
$\therefore \overline{DA}^2 : \overline{DC}^2 :: m : n$.

$\therefore \overline{DF}^2 : \overline{DE}^2 :: m : n$. (?)

$\therefore \overline{DF}^2 : s^2 :: m : n$. (?)

273. Proposition L.—Problem.

A polygon being given, to construct a similar polygon whose area shall be in a given ratio to the area of the given polygon.



Let P be the given polygon, one side of which is a, and $m : n$ the given ratio. Find a line a' such that

$a'^2 : a^2 :: m : n$ (**272**).

On a' as a side homologous to a, construct the polygon P' similar to P (**261**). Then, P' is the required polygon.

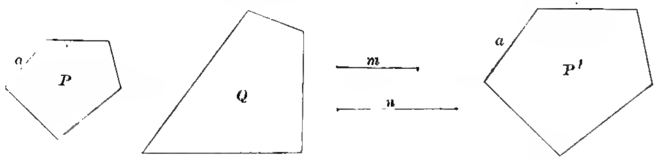
For, $P' : P :: a'^2 : a^2$ (**243**).

But, $a'^2 : a^2 :: m : n$.

$\therefore$ $P' : P :: m : n$. (?)

274. Proposition LI.—Problem.

To construct a polygon similar to a given polygon P, *and equivalent to a given polygon* Q.



Find m, n, sides of squares, such that $m^2 = P$, $n^2 = Q$ (**265**). Let a be a side of P; then, find a fourth proportional, a', to m, n, and a, and on a' as a side homologous to a, construct a polygon P' similar to P. Then will P' be the polygon required.

For, $m : n :: a : a'$.

$\therefore$ $m^2 : n^2 :: a^2 : a'^2$.

But, $P : Q :: m^2 : n^2$.

$\therefore$ $P : Q :: a^2 : a'^2$. (?)

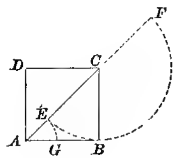
But, $P : P' :: a^2 : a'^2$ (**243**).

$\therefore$ $P : Q :: P : P'$; $\therefore$ $P' = Q$. (?)

275. Proposition LII.—Problem.

To prove that a side of a square and its diagonal are incommensurable.

The side CB is contained in the diagonal once, with a remainder AE, which is to be compared with CB, or with its equal AB. Since AB is perpendicular to the radius CB, AB is tangent to the arc EBF; and, since AF is a secant drawn from the same point A, and AE its external segment, we have



$$AE : AB :: AB : AF \quad (\mathbf{248}).$$

Therefore, instead of applying AE to AB, we can apply AB to AF. But AB is contained in AF twice, with a remainder AE, which is again to be applied to AB, instead of which we may again apply AB to AF, and so on indefinitely; hence, the process will never terminate, and AB and AC have no common measure. Hence, the side of a square and the diagonal are incommensurable.

276. Exercises.

1. Draw a straight line at random and divide it into 4 parts proportional to 2, 4, 3, 5 (**251**).

2. Divide a given straight line into 3 equal parts (**252**).

3. Draw three lines and find their fourth proportional (**253**).

4. Draw two lines and find their third proportional (**254**).

5. Given 2 *in.* : x :: x : 3 *in.*, to construct x (**255**).

6. Given $x = \sqrt{10}$, $y = \sqrt{7}$, to construct x and y (**255**).

7. Divide a line 2 in. long, internally and externally in extreme and mean ratio (**256**).

8. Divide a line 3 in. long, harmonically in the ratio of 5 to 3 (**258**).

9. Construct the locus of all the points whose distances from the extremities of a line 2 in. long are in the ratio of 2 : 1 (**260**).

10. Draw a pentagon and construct a similar pentagon such that each side of the latter shall be to the homologous side of the former as 3 to 2 (**262**).

11. Construct a square equivalent to a given pentagon (**265**).

12. Construct a square equivalent to the sum of the squares whose sides are respectively 3 in., 5 in., 6 in. (**266,** 1).

13. Construct a square equivalent to the difference of the squares whose sides are respectively 5 in. and 3 in. (**266,** 2).

14. Construct a square equivalent to the sum of a given rectangle and a given pentagon (**267**).

15. On a line 4 in. long construct a rectangle equivalent to the rectangle whose adjacent sides are 2 in. and 5 in., respectively (**268**).

16. Construct a rectangle whose area shall be 4 sq. in., and the sum of two of whose adjacent sides shall be 6 in. (**269**); also, a rectangle having the same area, the difference of two of whose adjacent sides is 3 in. (**270**).

17. Given a pentagon, to construct a similar pentagon whose area shall be to that of the given pentagon as 5 to 3 (**273**).

18. Construct a quadrilateral similar to a given quadrilateral and equivalent to a given pentagon (**274**).

19. Given $x + y = s$, $x^2 + y^2 = h^2$, to construct x and y.

20. Divide a given line into three segments, a, b, c, such that $a : b :: m : n$ and $b : c :: p : q$ (**251**).

21. Construct the locus of a point whose distances from two given parallel straight lines are in a given ratio.

22. Construct the locus of a point whose distances from two given non-parallel straight lines are in a given ratio.

23. Through a given point draw a straight line so that the distances from two given points to this line shall be in a given ratio.

24. Construct the locus of a point which divides all the chords of a given circle into internal or external segments whose rectangle is constant.

BOOK IV.

I. THE CIRCLE AND REGULAR POLYGONS.

277. Proposition I.—Theorem.

An inscriptible equilateral polygon is regular.

Let an equilateral polygon of n sides be inscribed in a circle.

The equal sides are equal chords, and therefore divide the circumference into n equal arcs (**126**).

Each angle is measured by $\frac{1}{2}(n - 2)$ of the equal arcs (**153**); hence, the polygon is equiangular, and, since it is also equilateral, it is regular (**93,** 7).

278. Corollary.

A regular polygon of any number of sides is possible.

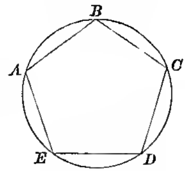
For, a straight line equal in length to any circumference can be divided into any number of equal parts (**252**); hence, the circumference can be divided into any number of equal arcs.

The chords of these equal arcs are equal (**126**); therefore, these chords are the sides of an inscribed equilateral polygon; but an inscribed equilateral polygon is regular (**277**).

Hence, a regular polygon of any number of sides is possible.

279. Proposition II.—Theorem.

An inscriptible equiangular polygon of an odd number of sides is regular.



Let the equiangular polygon $ABCDE$, of an odd number of sides, be inscribed in a circle.

The equal inscribed angles intercept equal arcs (**154**, 7). If each of these arcs be subtracted from the whole circumference, the remaining arcs will be equal; hence, the arc ABC = the arc BCD. Subtracting the common arc BC, we have the arc AB = the arc CD; that is, the first of any three consecutive arcs is equal to the third.

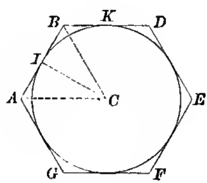
$\therefore$ arc AB = arc CD = arc EA = arc BC = arc DE.

Likewise, for any odd number of sides, the arcs are equal.

Hence, the sides of the polygon, which are the chords of these equal arcs, are equal (**126**). Therefore, the polygon is equilateral, and, since it is also equiangular, it is regular (**93**, 7).

280. Proposition III.—Theorem.

A circumscriptible equiangular polygon is regular.



Let $ABDEFG$ be an equiangular polygon circumscribed about a circle; I and K, points of tangency. Draw CA, CI, CB.

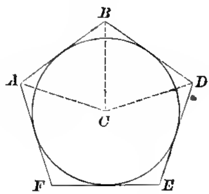
The angle CAI = the angle CBI (**176**, 1). The angles, AIC, BIC, are equal; (?) hence, the angles, ACI, BCI, are equal. (?) Therefore, the triangles, AIC, BIC, are equal; (?) $\therefore$ $AI = BI$; (?) $\therefore$ $AB = 2BI$.

Likewise, $BD = 2BK$; but, $BI = BK$; $\therefore$ $AB = BD$. In like manner, it can be proved that $BD = DE = \ldots$.

Hence, the polygon is regular. (?)

281. Proposition IV.—Theorem.

A circumscriptible equilateral polygon of an odd number of sides is regular.



Let $ABDEF$ be an equilateral polygon of an odd number of sides circumscribed about a circle. Draw CA, CB, CD.

The triangles, ABC, BDC, are equal (**55**); hence, the angles, BAC, BDC, are equal. But,

$BAF = 2BAC$, $BDE = 2BDC$; (?) $\therefore$ $BAF = BDE$.

In like manner, it can be proved that the first of any three consecutive angles is equal to the third.

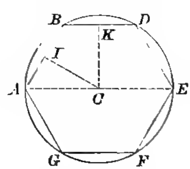
$$\therefore \quad A = D = F = B = E.$$

Hence, the polygon is regular. (?)

282. Proposition V.—Theorem.

Any regular polygon is inscriptible.

Let $ABDEFG$ be a regular polygon. Bisect AB, BD, by the perpendiculars, IC, KC. The intersection C of these perpendiculars is equally distant from A, B, D (**70**, 2).



Let $ABKC$ revolve about CK as an axis. Then, since the angles at K are right, and $KB = KD$, KB will coincide with KD. Since the angles, B, D, are equal,

also the sides, BA, DE (**93,** 7), BA will coincide with DE, and hence, CA with CE.

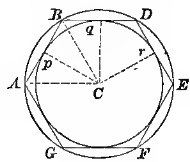
Therefore, the circumference described about C as a center, with CA as a radius, will pass through the vertices, A, B, D, E.

In like manner, it can be proved that this circumference will pass through the vertices, F, G. Hence, the polygon is inscriptible.

283. Proposition VI.—Theorem.

Any regular polygon is circumscriptible.

Let $ABDEFG$ be a regular polygon. By the last proposition, any regular polygon is inscriptible, and, being inscribed, its equal sides are equal chords of the circle, and are therefore equally distant from the center. If, then, a circumference be described about this center, with a radius equal to its distance from the sides, it will be tangent to the sides, and the polygon will be circumscribed about the circle.



284. Definitions and Corollaries.

1. The **center** of a regular polygon is the common center of the inscribed and circumscribed circles.

2. The **radius** of a regular polygon is the radius of the circumscribed circle.

3. The **apothem** of a regular polygon is the radius of the inscribed circle.

4. The **central angle** of a regular polygon is the angle included by the two radii drawn to the extremities of the same side.

5. *The straight lines joining the center and vertices of a regular polygon bisect the angles of the polygon.* (?)

6. *The central angles of a regular polygon are all equal, and each is equal to four right angles divided by the number of sides, and is the supplement of any angle of the polygon.* (?)

7. *A regular circumscribed polygon divides the circumference into equal parts at the points of tangency.* (?)

8. *A regular inscribed polygon being given, a regular inscribed polygon of double the number of sides can be constructed by bisecting the arcs subtended by the sides, and drawing chords to the arcs thus formed.* (?)

285. Proposition VII.—Theorem.

Regular polygons of the same number of sides are similar.

The polygons are mutually equiangular (**99**, 1), and their homologous sides are proportional; (?) hence, the polygons are similar.

286. Corollaries.

1. *The perimeters of regular polygons of the same number of sides are proportional to their radii or to their apothems* (**236**).

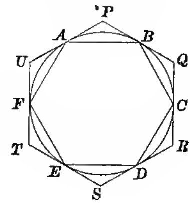
2. *The areas of regular polygons of the same number of sides are proportional to the squares of their radii or to the squares of their apothems* (**244**, 1).

287. Proposition VIII.—Problem.

A regular inscribed polygon being given, to circumscribe a similar polygon about the circle, and conversely.

1. To construct the circumscribed polygon.

1*st method.* Let $ABCD\ldots$ be a regular inscribed polygon. Draw tangents at the vertices, A, B, $C,\ldots$ intersecting in the points P, Q, $R,\ldots$ Then, $PQR\ldots$ is a regular circumscribed polygon similar to $ABCD\ldots$



For, these polygons have the same number of sides, since $PQR\ldots$ has a side for each angle of $ABC\ldots$

The triangles, APB, BQC, $CRD,\ldots$ are equal and isosceles; for, AB, BC, $CD,\ldots$ are equal, since they are sides of the regular inscribed polygon; and the angles, PAB, PBA, $QBC,\ldots$ are equal, since each is formed by a tangent and one of the equal chords, AB, BC, $CD,\ldots$ drawn to the point of tangency, and therefore measured by one-half of one of the equal arcs, AB, BC, $CD\ldots$

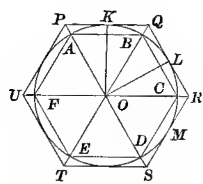
Hence, the angles, P, Q, $R,\ldots$ are equal, also the sides, AP, PB, $BQ,\ldots$ (**58**);

$$\therefore \quad PB + BQ = QC + CR = RD + DS = \ldots$$

$$\text{or,} \quad PQ = QR = RS = \ldots$$

Hence, the circumscribed polygon $PQR\ldots$ is both equilateral and equiangular, and therefore regular (**93**, 7), and consequently similar to ABC (**285**).

2*d method.* It is evident from the construction in the first method, that if the circumference be divided into equal parts, the tangents drawn at the points of division will form a regular circumscribed polygon.



Hence, bisect the arcs, AB, $BC,\ldots$ and draw the tan-

gents, PQ, QR, ... Then, PQR ... will be the regular circumscribed polygon required; for, since the arcs, AB, BC, ... are equal, their halves, AK, KB, BL, ... are equal; hence, also, $KB + BL$, $LC + CM$, ... or KL, LM, ... are equal.

2. To construct the inscribed polygon.

1*st method.* Draw chords joining the consecutive points of tangency of the sides of the circumscribed polygon, as in the first figure. Then, since the arcs, AB, BC, ... are equal (**284,** 7), the chords, AB, BC, ... are equal, and ABC ... is the regular inscribed polygon required.

2*d method.* Draw lines joining the center with the vertices of the circumscribed polygon, and draw chords joining the consecutive points in which these lines intersect the circumference, as in the second figure. Then, $ABCD$ ... is the inscribed polygon required.

For, the lines, OP, OQ, ... bisect the equal angles of the circumscribed polygon (**284,** 5); hence, in the triangles, OPQ, OQR, ... the angles, POQ, QOR, ... are equal (**53,** 9); therefore, the arcs, AB, BC, ... are equal (**148**); hence, the chords, AB, BC, ... are equal (**126**); therefore, $ABCD$ ... is the regular inscribed polygon required (**277**).

In the second construction, the sides of the two polygons are respectively parallel, since they are perpendicular to the radii drawn to the points of tangency (**133**), (**123,** 3).

288. Corollaries.

1. *If chords be drawn from the vertices of a regular inscribed polygon to the adjacent points of tangency of the sides of the similar circumscribed polygon, in case the sides are parallel, a*

regular inscribed polygon will be formed of double the number of sides, its perimeter will be greater than the perimeter of the given inscribed polygon, and its area greater than the area of that polygon. (?)

2. *If tangents be drawn at the vertices of a regular inscribed polygon to the sides of the similar circumscribed polygon, in case the sides are parallel, a regular circumscribed polygon will be formed of double the number of sides, its perimeter will be less than the perimeter of the given circumscribed polygon, and its area less than the area of that polygon.* (?)

3. *The perimeter of any regular inscribed polygon is less than the circumference* (**68**), *and its area is less than the area of the circle* (**21**, 7).

4. *The perimeter of any regular circumscribed polygon is greater than the circumference, and its area is greater than the area of the circle.*

The perimeter is greater than the circumference; for, let successive regular circumscribed polygons be constructed, each having twice as many sides as the next preceding. The perimeter of each is less than that of the preceding, and approaches more nearly to an equality with the circumference, which is therefore less than any of these perimeters.

The area is greater than the area of the circle (**21**, 6).

289. Proposition IX.—Problem.

To inscribe a square in a given circle.

Draw two diameters at right angles, and join their extremities by chords. The figure thus formed is a square (**148**), (**126**), (**154**, 2).

290. Corollaries.

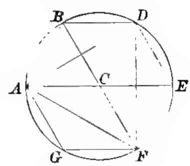
1. *By continually bisecting the arcs and drawing chords, we can construct regular inscribed polygons of* 8, 16, 32, ... *sides.*

2. *Regular circumscribed polygons of* 4, 8, 16, ... *sides can be constructed by* **287**.

291. Proposition X.—Problem.

To inscribe a regular hexagon in a given circle.

Let $ABDEFG$ be a regular inscribed hexagon. Draw the diameters, AE, BF. Each of the angles, ACB, ABC, BAC, is measured by one of the equal divisions of the circumference (**151**), (**153**); hence, the triangle ABC is equiangular, and therefore equilateral. Therefore, the side of a regular inscribed hexagon is equal to the radius.



Hence, to inscribe a regular hexagon in a given circle, apply the radius, as a chord, six times to the circumference.

292. Corollaries.

1. *To inscribe an equilateral triangle in a given circle, draw chords joining the alternate vertices of the regular inscribed hexagon.* (?)

2. *To inscribe a regular dodecagon in a given circle, bisect the arcs subtended by the sides of the regular inscribed hexagon, and draw chords joining the points of bisection with the adjacent vertices of the hexagon.*

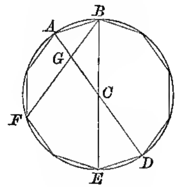
3. *In like manner, regular inscribed polygons of* 24, 48, 96, ... *sides can be constructed.*

4. *Regular circumscribed polygons of* 3, 6, 12, ... *sides can be constructed by* **287**.

293. Proposition XI.—Problem.

To inscribe a regular decagon in a given circle.

Suppose the construction made, and that each of the arcs, ABD, BDE, is five-tenths of the circumference. Then, AD, BE, are diameters, and intersect at the center C. Draw BF so that AF shall be two-tenths of the circumference.



The angles, ABC, BAC, AGB, are equal, since each is measured by two-tenths of the circumference (**153**), (**156**). Hence, the triangles, ABC, ABG, are isosceles (**61**) and similar (**227**, 1).

$$\therefore \quad BG = AB, \text{ and } AC : AB :: AB : AG.$$

The angles, CBG, BCG, are equal, since each is measured by one-tenth of the circumference (**153**), (**151**, 1).

$$\therefore \quad CG = BG, \text{ but } BG = AB, \ \therefore \ CG = AB.$$

Substituting CG for AB in the proportion, we have

$$AC : CG :: CG : AG.$$

Hence, the radius AC is divided in extreme and mean ratio at the point G (**250**, 1). Since $AB = CG$, we have the following construction:

Divide the radius in extreme and mean ratio (**256**); then apply the greater segment, as a chord, ten times to the circumference.

294. Corollaries.

1. *By continually bisecting the arcs and drawing chords, we can construct regular inscribed polygons of* 20, 40, 80, ... *sides.*

2. *By joining the alternate vertices of the decagon, we shall have a regular inscribed pentagon.*

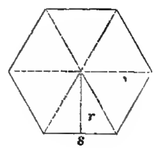
3. *Since* $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$, *the chord of the difference of one-sixth and one-tenth of the circumference is the side of a regular inscribed decapentagon, which can be constructed by applying this chord fifteen times to the circumference.*

The decapentagon being constructed, we can construct inscribed polygons of 30, 60, 120, ... *sides.*

4. *Regular circumscribed polygons of* 5, 10, 20, ... *also of* 15, 30, 60, ... *sides, can be constructed by* **287.**

295. Proposition XII.—Theorem.

The area of a regular polygon is equal to the product of its perimeter by one-half of its apothem.



Let s denote one side of the polygon, n the number of sides, p the perimeter, r the apothem, and P the area.

The radii of the polygon divide it into n triangles, each of which has s for its base, r for its altitude, and $s \times \frac{1}{2} r$ for its area; hence, the area of the n triangles is $ns \times \frac{1}{2} r$. But the area of the n triangles is the area of the polygon, and $ns = p$. $\therefore \quad P = p \times \frac{1}{2} r$.

296. Exercises.

1. An equiangular quadrilateral is inscriptible.

2. An equilateral quadrilateral is circumscriptible.

3. Is an inscribed equiangular polygon necessarily regular?

4. Is a circumscribed equilateral polygon necessarily regular?

5. If R denote the radius of a regular inscribed polygon, s one side, r the apothem, A one angle, and C the central angle, prove,

1st. That in case of a regular inscribed triangle, $s = R\sqrt{3}$, $r = \frac{1}{2}R$, $A = 60°$, $C = 120°$.

2d. In case of an inscribed square, $s = R\sqrt{2}$, $r = \frac{1}{2}R\sqrt{2}$, $A = 90°$, $C = 90°$.

3d. In case of a regular inscribed hexagon, $s = R$, $r = \frac{1}{2}R\sqrt{3}$, $A = 120°$, $C = 60°$.

4th. In case of a regular inscribed decagon, $s = R\frac{\sqrt{5}-1}{2}$, $r = \frac{1}{4}R\sqrt{10+2\sqrt{5}}$, $A = 144°$, $C = 36°$.

6. If s_3 denote one side of a regular inscribed triangle, s_4 one side of an inscribed square, prove that

$$s_3 : s_4 :: \sqrt{3} : \sqrt{2}.$$

7. The area of the regular inscribed hexagon is three-fourths of the area of the regular circumscribed hexagon.

8. The area of a regular inscribed hexagon is a mean proportional between the areas of the regular inscribed and circumscribed triangles.

9. The area of a regular inscribed dodecagon is equal to three times the square of the radius.

10. If a circumscriptible quadrilateral has two opposite sides parallel, the line passing through the center, parallel to the parallel sides and terminated by the other sides, is one-fourth of the perimeter.

II. THEORY OF LIMITS.

297. Preliminaries.

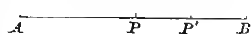
1. A **variable** is a quantity which admits of an indefinite number of successive values.

2. The **limit** of a variable is the constant which the variable approaches indefinitely.

3. *A variable never reaches its limit.* For, if a variable reach its limit, it could not approach the limit indefinitely.

4. *The difference between a variable and its limit is a variable whose limit is zero; and conversely, if the limit of the difference between a variable and a constant is zero, the constant is the limit of the variable.* For, since the variable approaches its limit indefinitely, the difference approaches zero indefinitely; and if the difference approaches zero indefinitely, the variable approaches the constant indefinitely.

5. Let a point move from A toward B: in the first second, one-half the distance from A to B, that is, to P; in the next second, one-half the distance from P to B, that is, to P'; and so on indefinitely. Then, the distance from A to the moving point is an increasing variable which indefinitely approaches the constant AB, as its limit, without ever reaching it; and the distance from the moving point to B is a decreasing variable which indefinitely approaches the constant zero, as its limit, without ever reaching it.



Let l be the symbol for limit; then, lx denotes the limit of x.

298. Proposition XIII.—Theorem.

If two variables are always equal, and each approaches a limit, these limits are equal.

Let x, y, be two variables, always equal; a, b, their respective limits.

Both variables are increasing or both decreasing. (?)

1st. Suppose the variables increasing. If a and b are not equal, one, as a, is the greater. Let $d = a - b$. Since x approaches a indefinitely, x may be made to differ from a by a quantity less than d.

$$\therefore \quad d > a - x, \quad \therefore \quad a - b > a - x, \quad \therefore \quad x > b.$$

Since the limit of y is b, and y is increasing, $y < b$, $\therefore$ $x > y$, which is contrary to the hypothesis; hence, a and b can not be unequal; $\therefore$ $a = b$, or $lx = ly$.

2d. Suppose the variables decreasing. If a and b are not equal, one, as a, is the greater. Let $d = a - b$. Since y approaches b indefinitely, y may be made to differ from b by a quantity less than d.

$$\therefore \quad y - b < d, \quad \therefore \quad y - b < a - b, \quad \therefore \quad y < a.$$

Since the limit of x is a, and x is decreasing, $x > a$, $\therefore$ $x > y$, which is contrary to the hypothesis; hence, a and b can not be unequal; $\therefore$ $a = b$, or $lx = ly$.

299. Proposition XIV.—Theorem.

If the limit of a variable is zero, the limit of the product of the variable by a constant is zero.

Let c be a constant, and x a variable whose limit is zero. As x approaches zero indefinitely, cx decreases indefinitely, and can be made to differ from zero by a quantity less than any assigned quantity; $\therefore$ $lcx = 0$ (**297**, 2).

300. Proposition XV.—Theorem.

The limit of the algebraic sum of two or more variables is the algebraic sum of their limits.

Let x, y, z, be variables, a, b, c, their respective limits, and u, v, w, the variable differences between x, y, z and a, b, c. Then, $x = a \mp u$, $y = b \mp v$, $z = c \mp w$, according as the variables are increasing or decreasing.

$$\therefore \quad x + y + z = a + b + c \mp u \mp v \mp w.$$

$$\therefore \quad l(x + y + z) = l(a + b + c \mp u \mp v \mp w) \quad (\mathbf{298}).$$

But, $\quad lx = a, \quad ly = b, \quad lz = c.$

$$\therefore \quad lu = 0, \quad lv = 0, \quad lw = 0 \quad (\mathbf{297},\ 4).$$

$$\therefore \quad l(a + b + c \mp u \mp v \mp w) = a + b + c \quad (\mathbf{297},\ 2).$$

$$\therefore \quad l(x + y + z) = a + b + c = lx + ly + lz.$$

301. Proposition XVI.—Theorem.

The limit of the product of a constant and a variable is the product of the constant and the limit of the variable.

Let c be a constant, and x a variable whose limit is a, and u the variable difference between x and a.

Then, $\quad x = a \mp u, \quad \therefore \quad cx = ca \mp cu.$

$$\therefore \quad lcx = l(ca \mp cu).$$

Since $\quad lx = a, \quad lu = 0 \quad (\mathbf{297},\ 4), \quad \therefore \quad lcu = 0 \quad (\mathbf{299}).$

$$\therefore \quad l(ca \mp cu) = ca, \quad \therefore \quad lcx = ca = clx.$$

302. Proposition XVII.—Theorem.

If the limit of each of two or more variables is zero, the limit of their product is zero.

Let the limit of each of the variables, x, y, z, ... be zero. Since each of the variables, x, y, z, ... approaches zero indefinitely, their product, $xyz\ldots$, approaches zero indefinitely.

$$\therefore \quad lxyz\ldots = 0 \quad (\mathbf{297}, 2).$$

303. Proposition XVIII.—Theorem.

The limit of the product of any number of variables is the product of their limits.

Let x, y, z, ... be variables, a, b, c, ... their respective limits, and u, v, w, ... the variable differences between x, y, z, ... and their respective limits.

$$\therefore \quad x = a \mp u, \qquad y = b \mp v, \qquad z = c \mp w \ldots$$

$$\therefore \quad xyz\ldots = abc\ldots \mp \text{ terms whose limits are zero.}$$

$$\therefore \quad lxyz\ldots = abc\ldots = lx\,.\,ly\,.\,lz\ldots$$

304. Corollaries.

1. *The limit of any power of a variable is the same power of its limit.*

For, $\quad lx^n = l\,(x\,.\,x\,.\,x\ldots) = lx\,.\,lx\,.\,lx\ldots = (lx)^n$.

2. *The limit of any root of a variable is the same root of its limit.*

For, $\quad l\,(\sqrt[n]{x})^n = (l\,\sqrt[n]{x})^n$; but, $\quad l\,(\sqrt[n]{x})^n = lx$.

$$\therefore \quad (l\,\sqrt[n]{x})^n = lx; \quad \therefore \quad l\,\sqrt[n]{x} = \sqrt[n]{lx}.$$

3. *The limit of the quotient of two variables is the quotient of their limits.*

Let $\quad z = \dfrac{x}{y}, \quad \therefore \quad yz = x, \quad \therefore \quad lyz = lx,$

$$\therefore \quad ly\,.\,lz = lx, \quad \therefore \quad lz = \frac{lx}{ly}, \quad \therefore \quad l\frac{x}{y} = \frac{lx}{ly}.$$

305. Proposition XIX.—Theorem.

If two variables are always in a constant ratio, and each approaches a limit, these limits are in the same ratio.

Let x and y be two variables whose constant ratio is r, and whose limits are respectively a and b.

Then, $\frac{x}{y} = r, \quad \therefore \quad x = ry.$

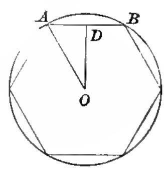
$\therefore \quad lx = lry$; but, $lx = a$, and, since $ly = b$,

$$lry = rb; \quad \therefore \quad a = rb, \quad \therefore \quad \frac{a}{b} = r, \quad \therefore \quad \frac{lx}{ly} = r.$$

III. MEASUREMENT OF THE CIRCLE.

306. Proposition XX.—Theorem.

If the number of sides of a regular inscribed polygon be increased indefinitely, the apothem will be an increasing variable whose limit is the radius.



Let $r = OA =$ the radius,

$r' = OD =$ the apothem.

$s = AB =$ one side.

In the triangle AOD, we have

$$r' < r \ (\mathbf{69},\ 1), \quad r - r' < \tfrac{1}{2}s \ (\mathbf{66}).$$

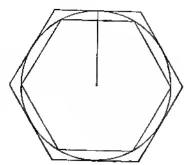
By increasing the number of sides indefinitely, s can be made less than any assigned quantity; for a stronger reason, $\frac{1}{2}s$ can be made less than any assigned quantity; and for a still stronger reason, $r - r'$ can be made less than any assigned quantity. $\therefore \ l(r - r') = 0, \ \therefore \ lr' = r$ (**297**, 4).

307. Proposition XXI.—Theorem.

If the number of sides of regular inscribed and circumscribed polygons be increased indefinitely, then,

1. *The circumference is the common limit of the perimeters of the polygons.*

2. *The area of the circle is the common limit of the areas of the polygons.*



1. Let p, p', respectively, denote the perimeters of similar circumscribed and inscribed regular polygons, r, r', respectively, their apothems, and c the circumference of the circle.

Then, $p : p' :: r : r'$ (**286**, 1).

$\therefore \quad p - p' : p :: r - r' : r; \quad \therefore \quad p - p' = \frac{p}{r}(r - r').$

$\therefore \quad l(p - p') = l\frac{p}{r}(r - r') = l\frac{p}{r}\,.\,l(r - r').$ (?)

But, $l(r - r') = 0$ (**306**); $\therefore \quad l\frac{p}{r}\,.\,l(r - r') = 0.$

$\therefore \quad l(p - p') = 0. \quad \therefore \quad p$ and p' approach each other indefinitely; but, $p > c$ and $p' < c$ (**288**, 4, 3).

$\therefore \quad p - c < p - p'$, and $c - p' < p - p'$.

$\therefore \quad l(p - c) = 0,$ and $l(c - p') = 0.$

$\therefore \quad lp = c,$ and $lp' = c$ (**297**, 4).

2. Let P, P', respectively, denote the areas of two similar circumscribed and inscribed regular polygons, p, p', their

perimeters, r, r', their apothems, and C the area of the circle.

$$P = \tfrac{1}{2} pr, \text{ and } P' = \tfrac{1}{2} p'r' \quad (\mathbf{295}).$$

$$\therefore \quad P - P' = \tfrac{1}{2}(pr - p'r').$$

But, $\quad p : p' :: r : r', \quad \therefore \quad p'r - pr' = 0.$

$$\therefore \quad P - P' = \tfrac{1}{2}(pr + p'r - pr' - p'r').$$

$$\therefore \quad P - P' = \tfrac{1}{2}(p + p')\,r - \tfrac{1}{2}(p + p')\,r'.$$

$$\therefore \quad P - P' = \tfrac{1}{2}(p + p')\,(r - r').$$

But, $l\,(r - r') = 0, \quad \therefore \quad l\,[\tfrac{1}{2}(p + p')\,(r - r')] = 0.$

$\therefore \quad l\,(P - P') = 0, \quad \therefore \quad P$ and P' approach each other indefinitely; but $P > C$, and $P' < C$ (**288**, 4, 3).

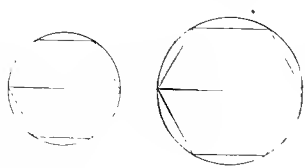
$$\therefore \quad P - C < P - P', \text{ and } \quad C - P' < P - P'.$$

$$\therefore \quad l\,(P - C) = 0, \qquad \text{and } l\,(C - P') = 0.$$

$$\therefore \qquad lP = C, \qquad \text{and} \qquad lP' = C \quad (\mathbf{297}, 4).$$

308. Proposition XXII.—Theorem.

The circumferences of two circles are proportional to their radii, and their areas are proportional to the squares of their radii.



Let r, r', respectively, be the radii of two circles, c, c', their circumferences, C, C', their areas, p, p', the perimeters of similar regular inscribed polygons, and P, P', their areas.

Then, $\dfrac{p}{p'} = \dfrac{r}{r'}$, and $\dfrac{P}{P'} = \dfrac{r^2}{r'^2}$ (**286**, 1, 2).

In these relations, which are true whatever be the number of sides, r, r', are constants, while p, p', P, P', are variables. If the number of sides be indefinitely increased, the perimeters approach the circumferences as their limits, and the areas of the polygons approach the areas of the circles as their limits (**307**).

That is, $lp = c$, $lp' = c'$, $lP = C$, $lP' = C'$.

$$\therefore \quad \frac{c}{c'} = \frac{r}{r'}, \text{ and } \frac{C}{C'} = \frac{r^2}{r'^2} \quad (\mathbf{305}).$$

309. Corollaries.

1. *The circumferences of two circles are proportional to their diameters, and their areas are proportional to the squares of their diameters, or to the squares of their circumferences.*

Let d, d', be the diameters.

Then, $$\frac{c}{c'} = \frac{r}{r'} = \frac{2r}{2r'} = \frac{d}{d'}.$$

Also, $$\frac{C}{C'} = \frac{r^2}{r'^2} = \frac{4r^2}{4r'^2} = \frac{d^2}{d'^2} = \frac{c^2}{c'^2}.$$

2. *The ratio of a circumference of a circle to its diameter, or to its radius, is constant; that is, the same for all circles.*

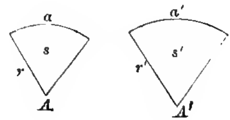
For, $\frac{c}{c'} = \frac{d}{d'}$, $\therefore \ \frac{c}{d} = \frac{c'}{d'}$, $\therefore \ \frac{c}{2r} = \frac{c'}{2r'}$, $\therefore \ \frac{c}{r} = \frac{c'}{r'}$.

The constant ratio of the circumference to the diameter is denoted by π.

Thus, $\frac{c}{d} = \pi$, $\therefore \ c = \pi d$, $\therefore \ c = 2\pi r$.

3. *Similar arcs are proportional to their radii, and similar sectors are proportional to the squares of their radii.*

Let a, a', denote the arcs; s, s', the sectors; r, r', the radii; A, A', the central angles; c, c', the circumferences; C, C', the areas of the circles; and R, a right angle. Then,



1. $\frac{a}{c} = \frac{A}{4R}$, and $\frac{a'}{c'} = \frac{A'}{4R}$ (**151**, 2).

But, $A = A'$ (**144**, 17), $\therefore \frac{a'}{c'} = \frac{a}{c}$, $\therefore \frac{c}{c'} = \frac{a}{a'}$.

But, $\frac{c}{c'} = \frac{r}{r'}$ (**308**), $\therefore \frac{a}{a'} = \frac{r}{r'}$.

2. $\frac{s}{C} = \frac{a}{c}$, and $\frac{s'}{C'} = \frac{a'}{c'}$ (**151**, 4).

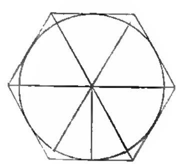
But, $\frac{a'}{c'} = \frac{a}{c}$, $\therefore \frac{s'}{C'} = \frac{s}{C}$, $\therefore \frac{C}{C'} = \frac{s}{s'}$.

But, $\frac{C}{C'} = \frac{r^2}{r'^2}$, (**308**), $\therefore \frac{s}{s'} = \frac{r^2}{r'^2}$.

310. Proposition XXIII.—Theorem.

The area of a circle is equal to the product of its circumference by one-half of its radius.

Let r denote the radius of the circle; c, its circumference; C, its area; r, the apothem of the circumscribed polygon; p, its perimeter; P, its area. Then,



$$P = p \times \tfrac{1}{2} r \quad (\mathbf{295}).$$

In this equation, which is true whatever be the number of sides, P, p, are variables, and r is a constant. If the number of sides be indefinitely increased, the perimeter approaches the circumference as its limit, and the area of the polygon approaches the area of the circle as its limit (**307**).

That is, $\quad lp = c, \quad$ and $\quad lP = C.$

$\therefore \quad C = c \times \frac{1}{2}r \quad$ (**301**).

311. Corollaries.

1. *The area of a circle is equal to the square of its radius multiplied by* π.

For, $\quad C = c \times \frac{1}{2}r, \quad$ and $\quad c = 2\pi r \quad$ (**309**, 2).

$\therefore \quad C = 2\pi r \times \frac{1}{2}r = \pi r^2.$

2. *The area of a sector is equal to the product of its arc by one-half of its radius.*

For, $\quad \frac{s}{C} = \frac{a}{c} \quad$ (**151**, 4), $\quad \therefore \quad \frac{s}{C} = \frac{a \times \frac{1}{2}r}{c \times \frac{1}{2}r}. \quad$ (?)

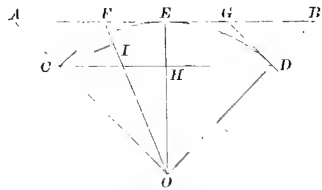
But, $\quad C = c \times \frac{1}{2}r, \quad \therefore \quad s = a \times \frac{1}{2}r. \quad$ (?)

312. Proposition XXIV.—Theorem.

Given the perimeter of a regular circumscribed polygon, and the perimeter of a similar inscribed polygon, to compute the perimeters of regular circumscribed and inscribed polygons of double the number of sides.

Let AB be a side of the given circumscribed polygon; CD, parallel to AB, a side of the similar inscribed polygon.

Then, E, the point of tangency, is the middle point of the arc CD (**135,** 2), and the chord CE is a side of the regular inscribed polygon of double the number of sides (**288,** 1).



Draw tangents at C and D, intersecting AB at F and G. Then, FG is a side of the regular circumscribed polygon of double the number of sides (**288,** 2).

Let p, p', denote the perimeters of the given circumscribed and inscribed polygons; p_d, p'_d, the perimeters of the regular circumscribed and inscribed polygons of double the number of sides.

Since OA is the radius of the circumscribed polygon,

$$\frac{p}{p'} = \frac{OA}{OC} = \frac{OA}{OE} \quad (\textbf{286,}\ 1).$$

Since OF bisects the angle AOE (**176,** 3), we have

$$\frac{OA}{OE} = \frac{AF}{FE} \quad (\textbf{221}), \quad \therefore \frac{p}{p'} = \frac{AF}{FE}.$$

This proportion taken by composition gives

$$\frac{p + p'}{p'} = \frac{AF + FE}{FE}.$$

Dividing by 2 and observing that $AF + FE = AE$, and that $2FE = FG$, we have

$$\frac{p + p'}{2p'} = \frac{AE}{FG}.$$

But AE is one-half of a side of the given circumscribed polygon whose perimeter is p; (?) FG is one side of the

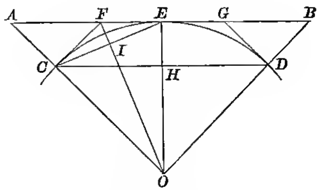
circumscribed polygon of double the number of sides whose perimeter is p_d. Hence, FG is contained the same number of times in p_d as AE is contained in p.

$$\therefore \quad \frac{p_d}{FG} = \frac{p}{AE}; \quad \therefore \quad \frac{AE}{FG} = \frac{p}{p_d}.$$

$$\therefore \quad \frac{p+p'}{2p'} = \frac{p}{p_d}; \quad \therefore \quad \frac{p+p'}{2p'} \times p_d = p.$$

$$\therefore \quad p_d = \frac{2p \times p'}{p+p'}.$$

The triangles, EFI, CEH, are similar, since the angles, I, H, are right angles; (?) and the alternate angles, FEI, ECH, are equal. (?)



$$\therefore \quad \frac{EI}{EF} = \frac{CH}{CE}.$$

Now, EF, EI, respectively, are halves of sides of polygons of the same number of sides whose perimeters are p_d, p'_d.

Hence, EF is contained the same number of times in p_d as EI is contained in p'_d.

$$\therefore \quad \frac{p_d}{EF} = \frac{p'_d}{EI}; \quad \therefore \quad \frac{EI}{EF} = \frac{p'_d}{p_d}.$$

Also, CH is one-half of a side of the given inscribed polygon whose perimeter is p', and CE is one side of the inscribed polygon of double the number of sides whose perimeter is p'_d.

Hence, CE is contained the same number of times in p'_d as CH is contained in p'.

$$\therefore \quad \frac{p'_d}{CE} = \frac{p'}{CH}; \quad \therefore \quad \frac{CH}{CE} = \frac{p'}{p'_d}.$$

$$\therefore \quad \frac{p'_d}{p_d} = \frac{p'}{p'_d}; \quad \therefore \quad (p'_d)^2 = p' \times p_d.$$

$$\therefore \quad p'_d = \sqrt{p' \times p_d}.$$

313. Proposition XXV.—Problem.

To compute the ratio of the circumference of a circle to its diameter.

Taking a circle whose diameter is 1, let p_4, p_8, $p_{16}, \ldots$ denote the perimeters of the regular circumscribed polygons of 4, 8, 16, ... sides, and p'_4, p'_8, $p'_{16}, \ldots$ the perimeters of the regular inscribed polygons of 4, 8, 16, ... sides.

Each side of the circumscribed square is 1, and each side of the inscribed square is $\frac{1}{2}\sqrt{2}$. (?)

$$\therefore \quad \begin{cases} p_4 = 4. \\ p'_4 = 4 \times \frac{1}{2}\sqrt{2} = 2.8284271. \end{cases}$$

$$\therefore \quad \begin{cases} p_8 = \dfrac{2p_4 \times p'_4}{p_4 + p'_4} = 3.3137085. \\ p'_8 = \sqrt{p'_4 \times p_8} = 3.0614675. \end{cases}$$

Continuing this process, we find

$$\begin{array}{llll}
p_{16} = 3.1825979, & p'_{16} = 3.1214452. \\
p_{32} = 3.1517249, & p'_{32} = 3.1365485. \\
p_{64} = 3.1441184, & p'_{64} = 3.1403312. \\
p_{128} = 3.1422236, & p'_{128} = 3.1412773. \\
p_{256} = 3.1417504, & p'_{256} = 3.1415138. \\
p_{512} = 3.1416321, & p'_{512} = 3.1415729. \\
p_{1024} = 3.1416025, & p'_{1024} = 3.1415877. \\
p_{2048} = 3.1415951, & p'_{2048} = 3.1415914. \\
p_{4096} = 3.1415933, & p'_{4096} = 3.1415923. \\
p_{8192} = 3.1415928, & p'_{8192} = 3.1415926.
\end{array}$$

The circumference of the circle whose diameter is 1 is less than 3.1415928, and greater than 3.1415926 (**288**, 3, 4), and is therefore 3.1415927, which is correct to the seventh decimal place.

$$\therefore \quad \pi = \frac{c}{d} = \frac{3.1415927}{1} = 3.1415927 \quad (\mathbf{309},\ 2).$$

For ordinary purposes, let $\pi = 3.1416$.

The above method of finding the ratio of the circumference to the diameter is called the *method of perimeters.*

314. Formulas for the Circle.

1. $r = \frac{1}{2}d.$	7. $c = 2\pi r.$
2. $r = \frac{c}{2\pi}.$	8. $c = \pi d.$
3. $r = \sqrt{\frac{C}{\pi}}.$	9. $c = 2\sqrt{\pi C}.$
4. $d = 2r.$	10. $C = \pi r^2.$
5. $d = \frac{c}{\pi}.$	11. $C = \frac{1}{4}\pi d^2.$
6. $d = 2\sqrt{\frac{C}{\pi}}.$	12. $C = \frac{c^2}{4\pi}.$

315. Exercises.

1. In circles of different radii, central angles subtended by arcs of equal length are inversely proportional to the radii.

2. Divide a given circle into a given number of equal parts by concentric circumferences.

3. Construct two circles whose radii are proportional to two given lines, and the sum of whose areas is equal to the area of a given circle.

4. In a given equilateral triangle, inscribe three circles, each tangent to two sides of the triangle and to the other two circles, and find their radii in terms of a side of the triangle.

5. Find the radii of three equal circles tangent to each other, and including between them one acre of land.

6. In a given circle inscribe three equal circles tangent to the given circle and to each other, and determine the radii of these equal circles in terms of the radius of the given circle.

7. If R, r, respectively denote the radius and apothem of a regular polygon, R', r', the radius and apothem of the isoperimetric polygon of double the number of sides, prove that

$$r' = \tfrac{1}{2}(R + r), \qquad R' = \sqrt{R \times r'}.$$

These formulas furnish a method of finding the ratio of the circumference to the diameter, called the *method of isoperimeters*.

IV. MAXIMA AND MINIMA.—SUPPLEMENTARY.

316. Definitions and Illustrations.

1. A **maximum** magnitude is the greatest of a class.

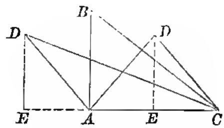
2. A **minimum** magnitude is the least of a class.

Thus, the diameter is the maximum chord of a circle. The perpendicular is the minimum line drawn from a given point to a given straight line.

317. Proposition XXVI.—Theorem.

Of all triangles having two sides respectively equal, that in which these sides include a right angle is the maximum.

Let BAC, DAC, be two triangles having the sides, BA, AC, respectively equal to the sides, DA, AC, and the angle BAC right, while the angle DAC is oblique; then, the area of BAC is greater than the area of DAC.



For, let fall the perpendicular DE upon the base or the base produced.

Then, $DA > DE$; (?) but $BA = DA$, $\therefore$ $BA > DE$.

$$\therefore \qquad \tfrac{1}{2} BA \times AC > \tfrac{1}{2} DE \times AC \quad (\textbf{23}, 14).$$

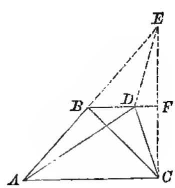
But, $BAC = \frac{1}{2} BA \times AC, \quad DAC = \frac{1}{2} DE \times AC$ (**199**).

$$\therefore \qquad BAC > DAC.$$

318. Proposition XXVII.—Theorem.

Of all triangles having equal areas and equal bases, that which is isosceles has the minimum perimeter.

Let ABC, ADC, be two triangles having equal areas and equal coincident bases, and let AB, BC, be equal, and AD, DC, unequal.



Since the areas are equal and the bases equal, the altitudes are equal. Since the altitudes are equal, the vertices, B, D, being on the same side of the base, are in a line parallel to the base.

Draw CF perpendicular to BD, and produce it to E in the prolongation of AB, and draw DE.

The angles, FBC, BCA, BAC, EBF, are equal (**46**, 2), (**59**), (**46**, 4). The right triangles, BFC, BFE, are equal (**57**).

$\therefore \quad BC = BE,\ \ FC = FE;$ (?) $\therefore\ DC = DE$ (**69**, 2).

Now, $AE < AD + DE$; $\therefore\ AB + BE < AD + DE$.

$\therefore \qquad AB + BC < AD + DC$ (**22**, 3).

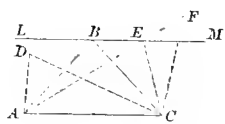
$\therefore \qquad AB + BC + CA < AD + DC + CA$ (**23**, 12).

319. Corollary.

Of all triangles having the same area, that which is equilateral has the minimum perimeter. (?)

320. Proposition XXVIII.—Theorem.

Of all triangles having equal perimeters and equal bases, that which is isosceles is the maximum.



Let ABC, ADC, be two triangles having equal perimeters and equal coincident bases, AB, BC, being equal, AD, DC, unequal. Then,

$$\text{area } ABC > \text{area } ADC.$$

The vertices, B, D, being on the same side of AC, D must be on LM, the parallel to AC through B, above LM, or below LM.

If D is on LM, at any point E, the triangles, ABC, AEC, having equal altitudes and equal bases, have equal areas. Then, $AB + BC + CA < AE + EC + CA$ (**318**), which is contrary to the hypothesis; $\therefore\ D$ can not be on LM.

If D is above LM, at any point F, $EC < EF + FC$.

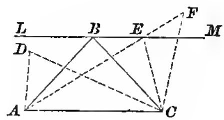
$\therefore\ AE + EC < AE + EF + FC$, or $AE + EC < AF + FC$,

$\therefore \qquad AE + EC + CA < AF + FC + CA$.

But, $\qquad AB + BC + CA < AE + EC + CA$,

$\therefore \qquad AB + BC + CA < AF + FC + CA$,

which is contrary to the hypothesis; $\therefore$ D can not be above LM.



Since D can be neither on LM nor above LM, D must be below LM; hence, the altitude of ABC is greater than the altitude of ADC; and, since the bases are equal, the area of ABC is greater than the area of ADC.

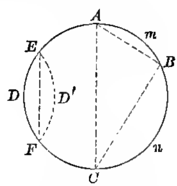
321. Corollary.

Of all isoperimetric triangles, that which is equilateral is the maximum. (?)

322. Proposition XXIX.—Theorem.

Of all isoperimetric plane figures, the circle is the maximum.

The area of a plane figure having a given perimeter may be indefinitely small, but can not be indefinitely great.



Let $ABCD$ be the maximum of all the isoperimetric plane figures having the given perimeter.

The maximum figure is convex; that is, any line, EF, joining any two points of the perimeter, lies within the figure; for, the area of the convex figure $ABCD$, is greater than that of the non-convex figure $ABCD'$, by the area of $EDFD'$.

Any straight line AC bisecting the perimeter bisects the area; for, if not, the symmetrical of the greater part can be substituted for the less, thus increasing the area of the maximum figure without increasing its perimeter, which is absurd.

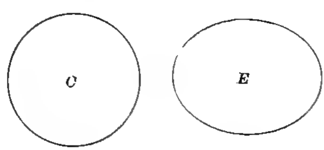
Assume any point B in the perimeter, and draw the straight lines, AB, BC. Then, the angle ABC is a right angle; for, if not, we may, without changing the lengths of the lines, AB, BC, or the portions of the perimeters, AmB, BnC, or the areas, ABm, BCn, make the angle ABC a right angle, thus increasing the area of the triangle ABC, and consequently, the area of the segment ACB, and since the segment ACD can be made equal to the segment ACB, thus increasing the maximum figure without increasing its perimeter, which is absurd.

Since B is any point in the semi-perimeter, ABC, estimated from A, any arbitrarily assumed point in the perimeter, and, since the angle ABC is always a right angle, this semi-perimeter is a semi-circumference (**179**, 3); hence, the whole perimeter is a circumference, and the figure is a circle.

323. Proposition XXX.—Theorem.

Of all plane figures having equal areas, the circle has the minimum perimeter

Let C be the area of a circle whose circumference is c, and E the area of an equivalent figure, not a circle, whose perimeter is p. Then, $c < p$.



For, $$c = p, \quad c > p, \quad \text{or} \quad c < p.$$

If $c = p$, then, $C > E$ (**322**), which is contrary to the hypothesis; therefore, c is not equal to p.

If $c > p$, C is greater than if $c = p$, since $C \propto c^2$ (**309,** 1). But if $c = p$, then, $C > E$; hence, if $c > p$, we have, for a stronger reason, $C > E$, which is contrary to the hypothesis; therefore, c is not greater than p.

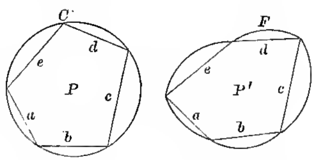
Now, neither $c = p$ nor $c > p$; $\therefore$ $c < p$.

324. Proposition XXXI.—Theorem.

Of all polygons having their sides respectively equal, that which is inscriptible in a circle is the maximum.

Let P, P', be polygons whose sides are a, b, c, d, e, respectively, P being inscriptible in a circle, and P' not inscriptible. Then,

$$P > P'.$$



Let a circle C be circumscribed about P, and on the sides, a, b, c, d, e, of the polygon P', let exterior circular segments, equal to those on the corresponding sides of P, be constructed.

The circle C and the irregular figure F are isoperimetric; hence, $C > F$ (**322**).

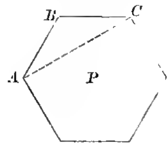
From each figure, subtracting its circular segments, we have

$$P > P'.$$

325. Proposition XXXII.—Theorem.

Of all isoperimetric polygons having the same number of sides, that which is regular is the maximum.

Let P be the maximum polygon having the given number of sides. Then, P is equilateral.



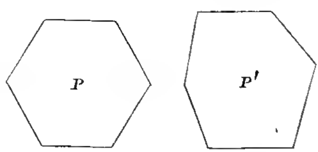
For, draw the diagonal AC. Then, the triangle ABC must be a maximum, in order that the polygon be a maximum. But ABC is a maximum when $AB = BC$ (**320**). Hence, any two consecutive sides of P are equal, and therefore P is equilateral.

Since P is the maximum polygon having given sides, it is inscriptible in a circle (**324**), and therefore regular (**277**).

326. Proposition XXXIII.—Theorem.

Of all polygons having equal areas and the same number of sides, the regular polygon has the minimum perimeter.

Let P be a regular polygon, and P' an irregular polygon, having equal areas and the same number of sides.



Let p be the perimeter of P, and p' the perimeter of P'. Then, $p < p'$.

For, $p = p'$, $p > p'$, or $p < p'$.

If $p = p'$, then, $P > P'$ (**325**), which is contrary to the hypothesis; hence, p is not equal to p'.

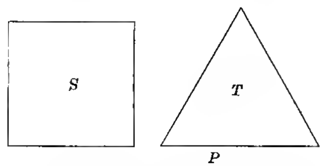
If $p > p'$, P is greater than if $p = p'$, since the area of P varies as the square of one side (**243**), and hence, as the square of its perimeter (**236**). But if $p = p'$, then, $P > P'$; hence, if $p > p'$, we have, for a stronger reason, $P > P'$, which is contrary to the hypothesis; hence, p is not greater than p'.

Now, neither $p = p'$ nor $p > p'$; $\therefore$ $p < p'$.

327. Proposition XXXIV.—Theorem.

Of any two isoperimetric regular polygons, that which has the greater number of sides has the greater area.

Let S be a square and T a regular triangle having an equal perimeter. Take any point P in one side of T, and consider it as dividing that side into two sides. Then, the triangle may be regarded as an irregular quadrilateral, while the square is regular. $\therefore$ $S > T$ (**325**).

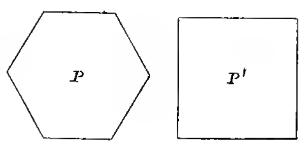


Also, it can be proved that a regular pentagon is greater than a square having an equal perimeter, and so on.

328. Proposition XXXV.—Theorem.

Of any two regular polygons having equal areas, that which has the greater number of sides has the less perimeter.

Let P and P' be two regular polygons having equal areas, P having a greater number of sides than P'.



Let p be the perimeter of P, and p' the perimeter of P'. Then, $p < p'$.

For, $\quad p = p', \quad p > p', \quad \text{or} \quad p < p'$.

If $p = p'$, then, $P > P'$ (**327**), which is contrary to the hypothesis; hence, p is not equal to p'.

If $p > p'$, P is greater than if $p = p'$, since $P \propto p^2$.

Therefore, if $p > p'$, we have $P > P'$, which is contrary to the hypothesis; hence, p is not greater than p'.

Now, neither $p = p'$ nor $p > p'$, $\therefore$ $p < p'$.

BOOK V.

I. LINES AND PLANES.

329. Definitions.

1. A **plane** is a surface such that a straight line joining any two of its points lies wholly in the surface. A plane, though indefinite in extent, is usually represented by a parallelogram.

2. The **foot** of a line is the point in which it meets a plane.

3. A straight line is *perpendicular to a plane*, if it is perpendicular to every straight line of the plane passing through its foot. In this case, the plane is perpendicular to the line.

4. The **distance** from a point to a plane is the perpendicular from the point to the plane.

5. A line is *parallel to a plane*, if all of its points are equally distant from the plane. In this case, the plane is parallel to the line.

6. A line is *oblique to a plane*, if it is neither perpendicular nor parallel to the plane. In this case, the plane is oblique to the line.

7. *Two planes are parallel*, if all the points of either are equally distant from the other.

8. The *projection of a point* on a plane is the foot of the perpendicular from the point to the plane.

9. The *projection of a line* on a plane is the locus of the projections of all its points.

10. The *angle* which a line makes with a plane is the angle which it makes with its projection on the plane. This angle is called the *inclination* of the line to the plane.

11. A *plane is determined* by lines or points, if no other plane can embrace these lines or points without being coincident with the first.

12. The *intersection of two planes* is the locus of all the points common to the two planes.

330. Proposition I.—Theorem.

An indefinite number of planes can embrace the same straight line.

In any plane draw a straight line, and move the plane till the line drawn in it coincides with the given line.

The plane revolving about the line as an axis takes an indefinite number of positions, each of which is the position of a plane embracing the line.

331. Proposition II.—Theorem.

A plane embracing a straight line and a point without that line, is determined in position.

For, let any plane embracing the line revolve about the line as an axis, till it embraces the point.

Now, if the plane revolve either way about the line as an axis, it will cease to embrace the point; hence, any

other plane embracing the line and point must coincide with the first plane, which is therefore determined (**329,** 11).

332. Corollaries.

1. *A plane embracing three points not in the same straight line is determined in position.*

For, the plane embraces the line joining any two of these points (**329,** 1), and the third point which is without that line (**331**).

2. *A plane embracing two intersecting straight lines is determined in position.*

For, this plane embraces one of the lines and any point of the other without the first.

3. *A plane embracing two parallels is determined in position.*

For, this plane embraces one of the parallels and any point of the other which is without the first.

4. *The intersection of two planes is a straight line.*

For, if any three points of the intersection were not in the same straight line, the two planes embracing them would coincide (**332,** 1), and therefore would not intersect.

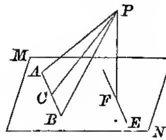
5. *A line embraced by each of two planes is their intersection.*

For, if not, the two planes embrace their intersection which is a straight line, and the points of the given line which are without the intersection, which is impossible (**331**).

333. Proposition III.—Theorem.

From any point without a plane, one perpendicular to the plane can be drawn, and only one.

Let P be the point and MN the plane. There can not be two minimum lines drawn from P to MN; for, if PA and PB are minimum lines, they are equal, the triangle APB is isosceles, and PC drawn from P to the middle point of AB is perpendicular to AB (**60**, 3); therefore,



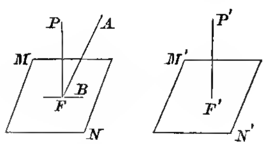
PA and PB are not perpendicular to AB (**30**); hence, PC is less than either PA or PB (**69**, 1), which are, therefore, not minimum lines; consequently, there can be only one minimum line from P to MN. Let PF be that minimum line; then, PF is perpendicular to any line, EF, drawn in the plane through its foot. For, if not, the perpendicular from P to EF would be less than PF; then, PF would not be the minimum line, which is contrary to the hypothesis; hence, PF is perpendicular to any and, consequently, to every line of the plane drawn through its foot, and therefore perpendicular to the plane (**329**, 3).

Since the minimum line and perpendicular are identical, and since there can be only one minimum line, there can be only one perpendicular.

334. Corollary.

At any point in a plane, one perpendicular to the plane can be erected, and only one.

Let F be the point in the plane MN. Let $P'F'$ be the perpendicular from P' to the plane $M'N'$ (**333**).



Now, let $M'N'$ be brought into coincidence with MN, so that F' shall coincide with F, $F'P'$ taking the position FP, which is therefore perpendic-

ular to MN at the point F, since it is perpendicular to the coincident plane.

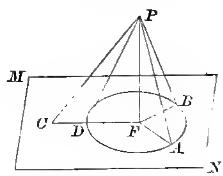
Only one perpendicular to MN can be erected at F; for, let FA be any other line through F, and let the plane PFA intersect MN in FB. Since PF is perpendicular to MN, it is perpendicular to FB (**329,** 3); hence, FA is not perpendicular to FB (**29,** 1), and therefore not perpendicular to the plane (**329,** 3).

335. Proposition IV.—Theorem.

If from a point without a plane a perpendicular and oblique lines be drawn,

1. *Oblique lines which meet the plane at equal distances from the foot of the perpendicular are equal.*

2. *Of two oblique lines which meet the plane at unequal distances from the foot of the perpendicular, the one which meets it at the greater distance is the greater.*



Let P be the point; MN, the plane; PF, the perpendicular; PA, PB, PC, oblique lines. Then,

1. If $FA = FB$, $PA = PB$ (**56,** 1).

2. If $FC > FA$, $PC > PA$; for, on FC take $FD = FA$, and draw PD. Then, $PD = PA$; but, $PC > PD$ (**69,** 3); $\therefore$ $PC > PA$.

336. Corollaries.

1. *Equal oblique lines from a point to a plane meet the plane at equal distances from the foot of the perpendicular from that point, and make equal angles with the perpendicular and also with the plane.* (?)

2. *Of two unequal oblique lines from a point to a plane, the greater meets the plane at the greater distance from the foot of the perpendicular, and makes the greater angle with the perpendicular and the less with the plane.* (?)

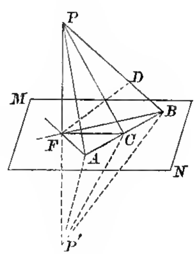
3. *Equal straight lines from a point to a plane meet the plane in the circumference of a circle whose center is the foot of the perpendicular from that point.* (?)

Hence, to draw a perpendicular from a point to a plane, draw any oblique line from the point to the plane; revolve this line about the point, tracing the circumference of a circle in the plane; draw a line from the point to the center of the circle; this line will be the perpendicular to the plane and the axis of the circle.

4. *The locus of all the points equally distant from the extremities of a line is the plane perpendicular to the line at its middle point.* (?)

337. Proposition V.—Theorem.

A straight line perpendicular to each of two straight lines at their intersection is perpendicular to their plane.



Let PF be perpendicular to FA, FB, at their intersection F; then, PF is perpendicular to their plane MN. For, let FC be any other straight line through F in the plane MN, and ACB any straight line intersecting FA, FC, FB, in A, C, B. Produce PF till $FP' = FP$, and draw PA, PB, PC, $P'A$, $P'B$, $P'C$.

Now, $AP = AP'$, $BP = BP'$ (**69**, 2); therefore, the triangles, APB, $AP'B$, are equal (**75**). Let the triangle $AP'B$ revolve about AB till it coincides with APB; then,

$P'C$, PC, coinciding, are equal; therefore, FC is perpendicular to PP' (**60**, 3), or PF is perpendicular to FC. Hence, PF is perpendicular to any, that is, to every line of MN through its foot, and consequently is perpendicular to the plane MN.

338. Corollaries.

1. *At any point in a straight line, one plane can be perpendicular to that line, and only one.*

For, the plane of the perpendiculars, FA, FB, to PP', is perpendicular to PP' (**337**).

Any other plane through F can not contain both A and B (**332**, 1), and must therefore intersect either PA or PB, say PB, in some other point D. Then, since PFB is a right angle, PFD is not a right angle; hence, PF is not perpendicular to FD, and therefore not perpendicular to the plane containing FD (**329**, 3).

2. *If one of two perpendiculars revolve about the other as an axis, maintaining its perpendicularity, it will generate a plane perpendicular to the axis, and this plane will be the locus of all the perpendiculars to the axis at the intersection of the two perpendiculars* (**329**, 3), (**338**, 1).

3. *Through any point without a straight line, one plane can be passed perpendicular to that line, and only one.*

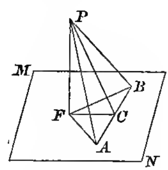
For, the plane generated by the revolution of the perpendicular from the point to the line, about the line as an axis, contains the point and is perpendicular to the line (**338**, 2).

If any other plane containing the given point intersect the line at the same point as the first, it is not perpendicular to the line (**338**, 1); and if it intersect the line at any

other point, the line joining the given point and the point of intersection is not perpendicular to the line (**30**); hence, the plane containing it is not perpendicular to the line (**329**, 3).

339. Proposition VI.—Theorem.

If from the foot of a perpendicular to a plane a straight line be drawn at right angles to any line of the plane, the line drawn from its intersection with the line of the plane to any point of the perpendicular, is perpendicular to the line of the plane.



Let PF be perpendicular to the plane MN, FC a perpendicular from the foot of PF to any line AB of MN; then, CP is perpendicular to AB.

For, take $CA = CB$, and draw FA, FB, PA, PB; then, $FA = FB$ (**69**, 2); $\therefore$ $PA = PB$ (**335**, 1); hence, PC is perpendicular to AB (**70**, 2).

340. Proposition VII.—Theorem.

A straight line and a plane are parallel if they can not meet, though both be produced indefinitely; and conversely.

For, if they are not parallel, all the points of the line would not be equally distant from the plane (**329**, 5); hence, the line would approach the plane in one direction or the other, and if sufficiently produced, would meet the plane, which is contrary to the hypothesis.

Conversely, if a line and plane are parallel, they can not meet. For, if they could meet, all the points of the line would not be equally distant from the plane, which is contrary to the definition (**329**, 5).

341. Proposition VIII.—Theorem.

Two planes are parallel if they can not meet, though both be produced indefinitely; and conversely.

For, if the planes are not parallel, all the points of one would not be equally distant from the other; hence, the planes would approach, and if sufficiently produced, would meet, which is contrary to the hypothesis.

Conversely, two parallel planes can not meet. For, if they could meet, all the points of one would not be equally distant from the other, which is contrary to the definition (**329**, 7).

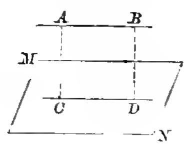
342. Corollary.

A line in one of two parallel planes is parallel to the other plane.

For, since the planes can not meet (**341**), any line in one can not meet the other, and is therefore parallel to it (**340**).

343. Proposition IX.—Theorem.

Either of two parallel lines is parallel to every plane embracing the other.



Let AB, CD, be parallel lines, and MN any plane embracing CD; then, AB is parallel to MN.

For, the parallels, AB, CD, are in the same plane $ABDC$ (**37**), (**332**, 3). The plane $ABDC$ intersects MN in CD (**332**, 5): hence, if AB meet MN, it must meet it in CD (**329**, 12). But AB can not meet CD (**38**); therefore, AB can not meet MN; hence, AB is parallel to MN (**340**).

344. Corollaries.

1. *Through any straight line a plane can be passed parallel to any other straight line.*

For, the plane of the first line and a parallel to the second through any point of the first, is parallel to the second line (**343**).

2. *Through any point a plane can be passed parallel to any two straight lines.*

For, the plane of the lines through the point, respectively parallel to the given lines, is parallel to each of the given lines (**343**).

345. Proposition X.—Theorem.

Planes perpendicular to the same straight line are parallel.

For, if the planes are not parallel, they would meet; then, there would be two planes through a point of their intersection perpendicular to the same straight line, which is impossible (**338,** 3); hence, the planes are parallel.

346. Proposition XI.—Theorem.

The intersections of two parallel planes with a third plane are parallel.

For, if these intersections are not parallel, they would meet (**39**); hence, the planes embracing them would meet, which is impossible (**341**).

347. Corollary.

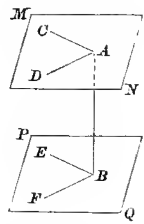
Parallel lines intercepted between parallel planes are equal.

For, the plane of the lines intersects the planes in parallel

lines (**346**) which, with the given parallel lines, form a parallelogram (**77**, 4) whose opposite sides are equal (**82**). Hence, the lines are equal.

348. Proposition XII.—Theorem.

A straight line perpendicular to one of two parallel planes is perpendicular to the other.



Let MN, PQ, be parallel planes, and let AB be perpendicular to PQ; then, AB is perpendicular to MN.

Let two planes embracing AB intersect MN, PQ, in AC, BE, and AD, BF; then, AC, BE, are parallel, also AD, BF (**346**); but AB is perpendicular to PQ, hence, to BE and BF (**329**, 3), hence, to AC and AD (**42**, 3), hence, to MN (**337**).

349. Corollary.

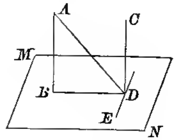
Through any point without a plane, one plane can be passed parallel to the given plane, and only one.

Let AB be a perpendicular from the point A to the plane PQ (**333**), and MN a plane through A perpendicular to AB (**338**, 1); then, MN is parallel to PQ (**345**). Now, any other plane through A is not perpendicular to AB (**338**, 1), and therefore not parallel to PQ (**348**).

350. Proposition XIII.—Theorem.

If one of two parallels is perpendicular to a plane, the other is perpendicular to the plane.

Let AB and CD be parallel, and AB perpendicular to MN; then, CD is perpendicular to MN.



For, BD in the plane MN is perpendicular to AB (**329,** 3), and therefore to CD (**42,** 3). Then, ED in the plane MN, perpendicular to BD, is perpendicular to AD (**339**), hence, to the plane of AD and BD (**337**), which is the plane of the parallels, and therefore to CD (**329,** 3). Hence, CD perpendicular to BD and ED at their intersection is perpendicular to their plane MN (**337**).

351. Corollaries.

1. *Two lines perpendicular to the same plane are parallel.*

For, a parallel to one of the perpendiculars through the foot of the other is perpendicular to the plane (**350**), and hence coincident with the given perpendicular at that point, since only one perpendicular can be erected to a plane at the same point (**334**).

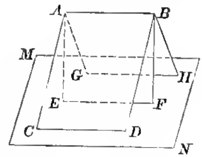
2. *Two straight lines parallel to a third straight line are parallel to each other.*

For, the two lines are perpendicular to any plane perpendicular to the third line (**350**), and hence are parallel to each other (**351,** 1).

352. Proposition XIV.—Theorem.

All the intersections of a plane with planes embracing a line parallel to the plane are parallel to that line and to one another.

Let CD, EF, GH, be the intersections of the plane MN with the planes AD, AF, AH, embracing the line AB parallel to MN; then, CD, EF, GH, are parallel to AB and to one another.



For, since AB is parallel to MN, it can not meet CD, which lies in the plane MN (**340**); and since AB and CD are in the same plane AD, and can not meet, they are parallel (**40**).

For like reason, AB is parallel to EF and to GH.

Hence, CD, EF, GH, are parallel to one another (**351,** 2).

353. Corollaries.

1. *A parallel to a line, through any point of a plane parallel to the line, lies in the plane.*

For, a parallel to AB through any point C of MN must coincide with CD, the intersection of MN with the plane embracing AB and the point C (**352**); otherwise there would be two lines through the point C parallel to AB, which is impossible (**42,** 2).

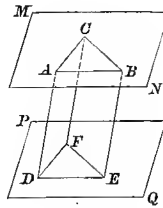
2. *The locus of any straight line through a given point parallel to a given plane is the plane through the point parallel to that plane.*

For, the intersection of the given plane with a plane embracing the line is parallel to the line (**352**), and to the plane through the point parallel to the given plane (**342**); hence, the line lies in that plane (**353,** 1).

3. *If each of two intersecting straight lines is parallel to a given plane, the plane of these lines is parallel to the given plane* (**353,** 2).

354. Proposition XV.—Theorem.

Two angles not in the same plane, having their sides respectively parallel and in the same direction, are equal, and their planes parallel.



Let the angles, A, D, in the planes, MN, PQ, have their sides, AB, AC, respectively parallel to DE, DF, and in the same direction.

The planes of AB, DE, and AC, DF, intersect in AD, and the plane BF, parallel to AD, intersects MN, PQ, in BC, EF, and AE, AF, in BE, CF. Then, BE, CF, are parallel to AD and to each other (**352**); hence, AE, AF, are parallelograms (**77**, 4); $\therefore$ $AB = DE$, $AC = DF$, $BE = AD$, $CF = AD$ (**82**); $\therefore$ $BE = CF$; $\therefore$ BF is a parallelogram (**86**); $\therefore$ $BC = EF$. Hence, the triangles, ABC, DEF, are equal (**75**); $\therefore$ the angles, BAC, EDF, are equal (**56**, 2).

Since AB, AC, are respectively parallel to DE, DF, they are parallel to PQ, the plane of DE, DF (**343**); hence, MN, the plane of AB, AC, is parallel to PQ (**353**, 3).

355. Corollaries.

1. *The angles formed by the intersections of two intersecting planes with parallel planes cutting the intersection of the two planes, are equal.*

For, MN, PQ, being parallel, AB, DE, are parallel; also, AC, DF (**346**); therefore the angles, BAC, EDF, are equal (**354**).

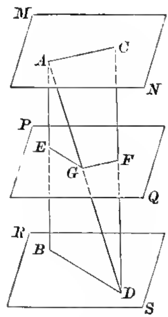
2. *The triangles formed by joining the corresponding extremi-*

ties of three equal parallel straight lines not in the same plane, are equal, and their planes parallel.

For, if AD, BE, CF, are equal and parallel, AE, AF, BF, are parallelograms; therefore, AB, DE, are equal and parallel; also, AC, DF, and BC, EF; hence, the triangles, ABC, DEF, are equal (**75**), and their planes are parallel (**354**).

356. Proposition XVI.—Theorem.

The corresponding segments of straight lines intersected by parallel planes are proportional.



Let the straight lines, AB, CD, be intersected by the parallel planes, MN, PQ, RS, in the points, A, E, B, and C, F, D. Then, $AE : EB :: CF : FD$.

For, let G be the point in which AD pierces PQ; then, EG, FG, the intersections of the planes, ABD, ACD, with PQ, are respectively parallel to BD, AC (**346**).

$$\text{Then, } \left\{ \begin{array}{l} AE : EB :: AG : GD, \\ AG : GD :: CF : FD, \end{array} \right\} \therefore\ AE : EB :: CF : FD.$$

357. Exercises.

1. Do two lines that are not parallel always meet?

2. Designate any three points, and show how a plane can be conceived to embrace them.

3. Why does a three-legged stool stand firm on a floor, when a four-legged chair may not?

4. Show how to find a perpendicular to the floor from a given point in the ceiling (**336**, 3).

5. Find a point in a plane equally distant from three given points not in the plane.

6. Through a given point pass a plane parallel to a given plane (**353**, 3).

7. Through a given point pass a plane perpendicular to a given straight line.

8. Through the point in which a given line pierces a given plane draw a line in that plane making a given angle with the given line.

9. Through a given point draw a straight line which shall intersect two given straight lines not in the same plane.

10. Through a given point draw, to a given plane, a straight line at a given inclination to the plane and parallel to another given plane.

11. Find the locus of the points in a given plane equally distant from two given points out of the plane.

12. Draw, from one of two given straight lines not in the same plane to the other, a straight line making a given angle with the first.

13. Through a given point draw a straight line intersecting the circumference of a circle and a given straight line not in the plane of the circle.

14. To determine that point in a given straight line which is equidistant from two given points not in the same plane with the given line.

15. Through a given point draw to a given plane a straight line of given length parallel to another given plane.

16. Find the locus of the points equidistant from three given points.

17. From the first to the second of three non-parallel straight lines not in the same plane draw a line parallel to the third.

18. Through a given point of a straight line parallel to a given plane draw a straight line of a given length terminating in the plane and making a given angle with the given line.

19. To find a point in one line at a given distance from another line not in the same plane.

20. Through a given point in a plane draw a straight line in the plane which shall be at a given distance from a given point without the plane.

II. SOLID ANGLES.

358. Definitions.

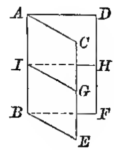
1. A **solid angle** is the divergence of two planes intersecting in a common line, or of three or more planes intersecting in a common point.

2. Solid angles are classified as *dihedral* and *polyhedral.*

3. A **dihedral angle** is the divergence of two intersecting planes.

4. The **faces** of a dihedral angle are the intersecting planes.

5. The **edge** of a dihedral angle is the intersection of its faces.



6. The **plane angle** of a dihedral angle is the plane angle formed by two straight lines, one in each face, perpendicular to the edge at the same point. Thus, the divergence of the planes, *AE*, *AF*, is a dihedral angle; *AE*, *AF*, are its faces; *AB* is its edge; *GIH* is its plane angle, if *GI*, *HI*, in the faces, are perpendicular to the edge *AB* at the same point *I*.

7. A dihedral angle is expressed by four letters, one in each face and two in the edge between the other two; thus, *CABD*, or *HIBE*. If, however, the dihedral angle is isolated, it may be expressed by two letters in its edge; thus, *AB*.

8. A dihedral angle may be *generated* by the revolution of one of two planes about a common straight line from a position of coincidence with the other.

9. **Adjacent dihedral angles** are those which have a common edge and a common face between them.

10. If the faces of a dihedral angle be produced through the edge, the angles on opposite sides of one plane and on the same side of the other are *adjacent*; the angles on opposite sides of both planes are *vertical*.

11. Dihedral angles are classified as *right* and *oblique*, and the oblique as *acute* and *obtuse*, as the analogous plane angles (**6**), the faces of the dihedral angle taking the place of the sides of the plane angle, and the edge, of the vertex.

12. Two planes are perpendicular to each other, or oblique, according as their dihedral angle is right or oblique.

13. When two planes are cut by a third plane in different lines, the dihedral angles formed take the same names as the analogous plane angles formed by one straight line intersecting two others (**36**, 2).

14. The *angle of two straight lines not in the same plane* is equal to the angle formed by two straight lines through the same point, respectively parallel to the given lines.

15. A **polyhedral angle** is the divergence of three or more planes meeting in a common point.

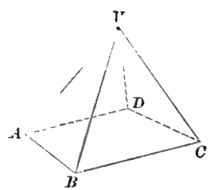
16. The **faces** of a polyhedral angle are the portions of the bounding planes limited by their intersections.

17. The **edges** of a polyhedral angle are the intersections of its faces.

18. The **vertex** of a polyhedral angle is the intersection of its edges.

19. The **facial angles** of a polyhedral angle are the plane angles formed by consecutive edges.

20. The **dihedral angles** of a polyhedral angle are the dihedral angles formed by consecutive faces.



Thus, $V\text{-}ABCD$ is a polyhedral angle; VAB, VBC, ... are its faces; VA, VB, ... its edges; V is its vertex; AVB, BVC, ... are its facial angles; $AVBC$, $BVCD$, ... its dihedral angles.

21. Polyhedral angles are classified as *trihedral*, *quadrihedral*, ... according to the number of faces.

22. Trihedral angles are *rectangular*, *birectangular*, or *trirectangular*, according as they have one, two, or three right dihedral angles.

23. Trihedral angles are *scalene, isosceles,* or *equilateral,* according as the facial angles are all unequal, two equal, or three equal.

24. Two trihedral angles are *supplementary* if the facial angles of one are respectively the supplements of the dihedral angles of the other.

25. A polyhedral angle is *convex* if the polygon formed by the intersections of a plane with all its faces is convex (**93**, 11).

26. Two polyhedral angles are *symmetrical* if they have the same number of faces, and the successive dihedral and facial angles respectively equal, but arranged in a reverse order.

359. Proposition XVII.—Theorem.

All the plane angles of a dihedral angle are equal.

For, since the corresponding sides of these plane angles are in the same plane, and perpendicular to the same line, the edge, they are parallel (**41**); hence, these plane angles are equal (**354**).

360. Corollary.

Two dihedral angles are equal if their plane angles are equal.

For, bringing their edges and plane angles into coincidence, the dihedral angles will coincide, and hence are equal (**332**, 2), (**21**, 5).

361. Proposition XVIII.—Theorem.

Two dihedral angles are proportional to their plane angles.

1. If the plane angles of the dihedral angles are commensurable, suppose lines to be drawn through their vertices

in their planes, dividing them into partial plane angles, each equal to the common unit of measure.

The planes embracing the edges and the several lines of division will divide the dihedral angles into equal partial dihedral angles, since their plane angles are equal (**360**).

Each of the given dihedral angles having one of the partial dihedral angles as a unit of measure, has the same numerical value as its plane angle, since it contains its unit of measure the same number of times.

The dihedral angles are proportional to their numerical values (**145**); the plane angles are also proportional to their numerical values; but the numerical values of the dihedral angles are equal to the numerical values of their corresponding plane angles; hence, the dihedral angles are proportional to their plane angles.

2. If the plane angles of the dihedral angles are incommensurable, the proposition can be proved by the method employed in **150,** 2.

362. Corollaries.

1. *The plane angle of a dihedral angle is the measure of the dihedral angle.* (?)

2. *If the faces of a dihedral angle be produced through the edge, the adjacent angles are supplemental and the vertical angles are equal.* (?)

3. *Two dihedral angles are equal in the following cases:*

1*st. If their faces are respectively parallel, and lie in the same direction or in opposite directions from their edges* (**48**).

2*d. If their edges are parallel, and the faces of one respectively meet, at right angles, the faces of a supplemental angle of the other.* (?)

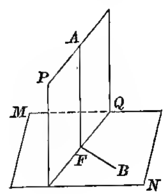
4. *Two dihedral angles are supplemental in the following cases:*

1st. *If two of their faces are parallel and lie in the same direction, and the other faces are parallel and lie in opposite directions from their edges.*

2d. *If their edges are parallel, and the faces of one respectively meet, at right angles, the faces of the other* (**49**).

363. Proposition XIX.—Theorem.

If a straight line is perpendicular to a plane, every plane embracing the line is perpendicular to that plane.



Let AF be perpendicular to the plane MN; then, any plane PQ, embracing AF, is perpendicular to MN.

For, at F draw, in the plane MN, FB, perpendicular to the intersection FQ.

Since AF is perpendicular to MN, it is perpendicular to FQ and FB. (?) Hence, the angle AFB is the plane angle (**358,** 6), and consequently, the measure of the dihedral angle of MN and PQ (**362,** 1); and since AFB is a right angle, the dihedral angle is a right angle; that is, PQ is perpendicular to MN.

364. Corollaries.

1. *If three straight lines are perpendicular to each other at a common point, each is perpendicular to the plane of the other two, and the three planes are perpendicular to each other.* (?)

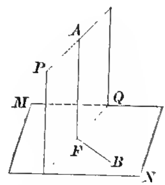
2. *A plane embracing a given line can be passed perpendicular to a given plane.*

For, the plane embracing both the given line and a

perpendicular from any one of its points to the given plane, is perpendicular to that plane (**363**).

365. Proposition XX.—Theorem.

If two planes are perpendicular to each other, a straight line in one, perpendicular to their intersection, is perpendicular to the other.



Let the planes, MN, PQ, be perpendicular to each other; then, AF in PQ, perpendicular to their intersection FQ, is perpendicular to MN.

For, if FB in MN is perpendicular to FQ, the angle AFB is right, since it is the measure of the right dihedral angle of the two planes; hence, AF, perpendicular to FQ and FB at their intersection, is perpendicular to their plane MN (**337**).

366. Corollaries.

1. *If two planes are perpendicular to each other, a straight line through any point of one perpendicular to the other lies in the first.*

For, a straight line in the first plane, through that point, perpendicular to the intersection, is the perpendicular to the second plane (**365**), (**333**), (**334**).

2. *If a straight line is oblique or parallel to a plane, only one plane embracing the line can be perpendicular to the plane.*

For, one such plane can be drawn (**364**, 2); but only one, since this plane must contain any perpendicular from the line to the given plane (**366**, 1), and only one plane can embrace two intersecting lines (**332**, 2).

3. *The projection of a straight line on a plane is a straight line.*

For, the plane embracing this line, perpendicular to the given plane, contains the perpendiculars from all the points of the line to the plane (**366,** 1); and its intersection with the given plane, which is a straight line (**332,** 4), is the locus of the projections of all the points of the line upon the plane, and hence the projection of the line.

The plane which embraces a given straight line, and is perpendicular to a given plane, is called the *projecting plane* of the line.

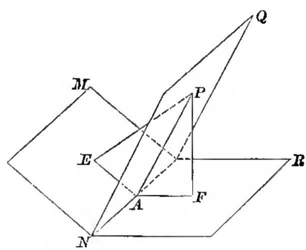
4. *If each of two intersecting planes is perpendicular to a third plane, their intersection is perpendicular to that plane.*

For, the perpendicular to the third plane, through any point of the intersection of the first and second, is in each of these planes (**366,** 1), and, hence, is their intersection.

Corollary 4 may be stated thus: *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*

367. Proposition XXI.—Theorem.

Every point in a plane bisecting a dihedral angle is equally distant from the faces of that angle.



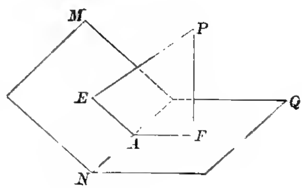
Let P be any point of the plane NQ, bisecting the dihedral angle, $MNAR$; then, the perpendiculars, PE, PF, from P to the faces, MN, NR, are equal.

The plane $PEAF$, of PE and PF, is perpendicular to the planes, MN, NR (**363**), and hence to their intersection

NA (**366,** 4); $\therefore$ NAE, NAP, NAF, are right angles (**329,** 3); hence, the plane angles, PAE, PAF, are respectively the measures of the dihedral angles, $MNAQ$, $QNAR$, which by hypothesis are equal; therefore their measures, PAE, PAF, are equal; hence the right triangles, PEA, PFA, are equal (**72,** 2); $\therefore$ $PE = PF$.

368. Proposition XXII.—Theorem.

The angle included by two perpendiculars drawn from any point within a dihedral angle to its faces, or the angle included by two lines drawn from any point, respectively parallel to these perpendiculars, in the same direction or in opposite directions, is the supplement of the dihedral angle.



Let PE, PF, be two perpendiculars drawn from the point P to the faces, MN, NQ, of the dihedral angle, $MNAQ$.

The plane of these perpendiculars is perpendicular to the faces (**363**), and, consequently, to the edge NA (**366,** 4); hence its intersections, AE, AF, with the faces, are perpendicular to the edge (**329,** 3); therefore EAF is the plane angle of the dihedral angle (**358,** 6), and hence, its measure (**362,** 1).

Since the angles, E, F, of the quadrilateral $PEAF$ are right, P is the supplement of A (**79**).

The angle included by two lines drawn from any point, respectively parallel to these perpendiculars, in the same direction or in opposite directions, is equal to the angle included by the perpendiculars, and hence is the supplement of the given dihedral angle.

369. Proposition XXIII.—Theorem.

The perpendiculars drawn from any point within a trihedral angle to its faces, or the lines drawn from any point respectively parallel to these perpendiculars, in the same direction or in opposite directions, are the edges of a trihedral angle which is the supplement of the given trihedral angle.

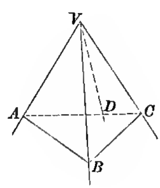
For, the facial angles formed by the perpendiculars are respectively the supplements of the dihedral angles of the given trihedral angle (**368**), (**358,** 24).

The lines drawn from any point respectively parallel to these perpendiculars, in the same direction or in opposite directions, form facial angles respectively equal to those formed by the perpendiculars, and hence are the supplements of the dihedral angles of the given trihedral angle (**358,** 24).

370. Proposition XXIV.—Theorem.

The sum of any two facial angles of a trihedral angle is greater than the third.

1. If either of two facial angles is greater than the third, their sum is greater than the third.



2. If each of the facial angles, AVB, BVC, of the trihedral angle, $V\text{-}ABC$, is less than the facial angle AVC, then,

$$AVB + BVC > AVC.$$

For, draw VD in the face AVC, making the angle AVD equal to the angle AVB. Through any points, A, C, of the edges, VA, VC, draw AC cutting VD in D; take $VB = VD$, and draw AB, BC.

The triangles, AVB, AVD, are equal (**55**); $\therefore\ AB = AD$ (**56**, 2). But,

$$AB + BC > AC \quad (\mathbf{65});$$

$$\therefore \quad AB + BC - AB > AC - AD;$$

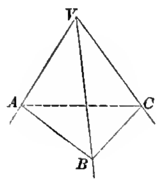
$\therefore \quad BC > DC$; $\therefore$ the angle $BVC >$ the angle DVC (**74**);

$$\therefore \quad AVB + BVC > AVD + DVC;$$

$$\therefore \quad AVB + BVC > AVC.$$

371. Proposition XXV.—Theorem.

The sum of the three facial angles of a trihedral angle is greater than zero and less than four right angles, but may have any value between these limits.



Let V-ABC be a trihedral angle.

1. Since each of the facial angles is greater than 0, their sum is greater than 0.

2. Join any three points, A, B, C, in the edges, thus forming the triangle ABC.

The sum of the angles of the three triangles whose common vertex is V, is equal to six right angles (**52**); and since the sum of the angles VAB, VAC, is greater than the angle BAC (**370**), and similarly for the angles about B and C, the sum of the angles at the bases of the same triangles is greater than the sum of the angles of the triangle ABC, that is, greater than two right angles; hence, the sum of the angles at the vertex V, which is the sum of the facial angles of the trihedral angle, is less than four right angles. This sum may have any value between 0° and 360°. (?)

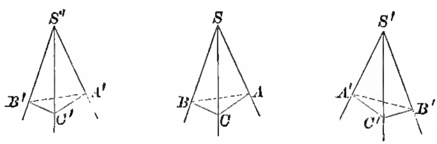
372. Corollary.

The sum of the three dihedral angles of a trihedral angle is greater than two and less than six right angles, but may have any value between these limits.

For, the sum of the dihedral angles of the given trihedral angle, plus the sum of the facial angles of its supplemental trihedral angle, is equal to six right angles (**358,** 24); but the sum of these facial angles is less than four right angles and greater than 0; hence, the sum of the dihedral angles of the given trihedral angle is greater than two and less than six right angles. This sum may have any value between 180° and 540°. (?)

373. Proposition XXVI.—Theorem.

Two trihedral angles, having the three facial angles of the one respectively equal to the three facial angles of the other, are equal if the equal facial angles are arranged in the same order; but are symmetrical and equivalent if the equal facial angles are arranged in a reverse order.



Let the trihedral angles, S, S', have the facial angles, ASB, $A'S'B'$, equal; also, ASC, $A'S'C'$, and BSC, $B'S'C'$. Then will the dihedral angles, SA, $S'A'$, be equal; also, SB, $S'B'$, and SC, $S'C'$.

Take any point A in the edge SA, and perpendicular to SA draw AB, AC, in the faces, ASB, ASC, respectively.

Take $S'A' = SA$, and perpendicular to $S'A'$ draw $A'B'$, $A'C'$, in the faces, $A'S'B'$, $A'S'C'$, respectively.

The two triangles, ASB, $A'S'B'$, are equal (**57**); therefore, $AB = A'B'$, $BS = B'S'$ (**56,** 2). For like reason, $AC = A'C'$, and $SC = S'C'$.

The two triangles, BSC, $B'S'C'$, are equal (**55**); therefore, $BC = B'C'$.

Hence the triangles, ABC, $A'B'C'$, are equal (**75**); therefore the angles, BAC, $B'A'C'$, are equal; but these plane angles are the measures of the dihedral angles, SA, $S'A'$ (**362,** 1), which are therefore equal. In like manner, it can be proved that the dihedral angles, SB, $S'B'$, are equal; also, SC, $S'C'$.

If the equal facial angles are arranged in the same order, as in the central and left-hand figures, the trihedral angles can be made to coincide, and are therefore equal; but if the equal facial angles are arranged in a reverse order, as in the central and right-hand figures, the trihedral angles are symmetrical and equivalent.

A familiar example of symmetrical equivalency is found in the two hands.

374. Exercises.

1. Pass a plane perpendicular to a given plane so as to embrace a given straight line.

2. Find the locus of points in space which are equally distant from two given points.

3. The angle which a straight line makes with its projection upon a plane is the least angle which it makes with any line of that plane.

4. A plane perpendicular to a line parallel to a plane is perpendicular to the plane.

5. Find the locus of the points which are equally distant from two given straight lines in the same plane.

6. The three planes bisecting the three dihedral angles of a trihedral angle intersect in the same straight line.

7. Find the locus of the points which are equally distant from three given planes.

8. The three planes embracing the edges of a trihedral angle, and respectively perpendicular to the opposite faces, intersect in the same straight line.

9. Find the locus of the points which are equally distant from three given straight lines in the same plane.

10. The three planes embracing the edges of a trihedral angle and the bisectors of the opposite facial angles respectively, intersect in the same straight line.

11. The three planes embracing the bisectors of the facial angles of a trihedral angle, and respectively perpendicular to the faces, intersect in the same straight line.

12. Find the locus of the points which are equally distant from the three edges of a trihedral angle.

13. Through a given point of a plane draw, in the plane, a straight line perpendicular to a given straight line in space.

14. Cut a given quadrahedral angle by a plane so that the section shall be a parallelogram.

15. Prove that a common perpendicular to any two lines in space can be drawn intersecting both; that only one such common perpendicular can be drawn; and that the common perpendicular is the shortest distance from one of these lines to the other.

16. The sum of the facial angles of any convex polyhedral angle is less than four right angles.

17. Two isosceles trihedral angles are equal if the three facial angles of the one are respectively equal to the three facial angles of the other.

18. The dihedral angles opposite the equal facial angles of an isosceles trihedral angle are equal.

19. A trihedral angle is isosceles if two of its dihedral angles are equal.

20. Two trihedral angles, having two facial angles and the included dihedral angle of one equal to two facial angles and the included dihedral angle of the other, are equal, or symmetrical and equivalent.

21. Two trihedral angles, having a facial angle and the two adjacent dihedral angles of one equal to a facial angle and the two adjacent dihedral angles of the other, are equal, or symmetrical and equivalent.

22. In two trihedral angles, having two facial angles of one respectively equal to two facial angles of the other, and the included dihedral angles unequal, the third facial angle belonging to the trihedral angle having the greater included dihedral angle is the greater.

23. In two trihedral angles, having two facial angles of one respectively equal to two facial angles of the other, and the third facial angles unequal, the dihedral angle opposite the greater third facial angle is the greater.

24. Two trihedral angles, having the three dihedral angles of one respectively equal to the three dihedral angles of the other, are equal if the equal dihedral angles are arranged in

the same order, but are symmetrical and equivalent if the equal dihedral angles are arranged in a reverse order.

25. All trirectangular trihedral angles are equal.

26. If the edges of any polyhedral angle be produced through the vertex, an equivalent polyhedral angle will be formed symmetrical with the given polyhedral angle.

27. In how many ways may three planes intersect one another? Draw illustrations, and show what kind of angles the planes form in each case.

28. Determine a point in a given plane such that the sum of its distances from two given points on the same side of the plane shall be a minimum.

29. Determine a point in a given plane such that the difference of its distances from two given points on opposite sides of the plane shall be a maximum.

30. Draw, in a given plane, through a given point in the plane, a straight line perpendicular to a given line not in that plane.

BOOK VI.

POLYHEDRONS.

375. General Definitions.

1. A **polyhedron** is a solid bounded by polygons.

2. The **faces** of a polyhedron are the bounding polygons.

3. The **edges** of a polyhedron are the intersections of its faces.

4. The **vertices** of a polyhedron are the intersections of its edges.

5. A **diagonal** of a polyhedron is a straight line joining any two vertices not in the same face.

6. A **section** of a polyhedron is the polygon formed by the intersection of a plane with three or more faces.

7. A **convex polyhedron** is a polyhedron every section of which is a convex polygon.

8. A **solid unit,** called also a *unit of volume,* is a given polyhedron assumed as the unit of measure for solids.

9. The **volume** of a polyhedron is its numerical value (**144,** 8).

10. **Similar polyhedrons** are polyhedrons identical in form.

11. **Equivalent polyhedrons** are polyhedrons identical in volume.

12. **Equal polyhedrons** are polyhedrons identical in form and volume.

13. A *tetrahedron* is a polyhedron of four faces; a *pentahedron* is a polyhedron of five faces; a *hexahedron* is a polyhedron of six faces.... Let all the polyhedrons, up to the icosahedron, be defined (**93,** 2).

I. PRISMS.

376. Definitions and Classification.

1. A **prism** is a polyhedron two of whose faces are equal and parallel polygons having their homologous sides parallel, and whose other faces are parallelograms having homologous sides of the equal polygons for bases.

2. The **bases** of a prism are the equal parallel faces.

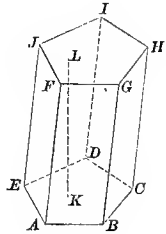
3. The **lateral faces** of a prism are all the faces except the bases.

4. The **lateral surface** of a prism is the sum of its lateral faces.

5. The **lateral edges** of a prism are the intersections of its lateral faces.

6. The **basal edges** of a prism are the intersections of the bases with the lateral faces.

7. The **altitude** of a prism is the perpendicular distance from one base to the plane of the other.



Thus, $ABCDE$-F is a prism; $ABCDE$, $FGHIJ$, are its bases; AG, BH, ... are its lateral faces, whose sum is the lateral surface; AF, BG, ... are its lateral edges; AB, BC, ... FG, GH, ... are its basal edges; KL, perpendicular to the bases, is the altitude.

8. A **right prism** is a prism whose lateral edges are perpendicular to its bases. Any lateral edge of a right prism is equal to the altitude. (?)

9. An **oblique prism** is a prism whose lateral edges are oblique to its bases. Any lateral edge of an oblique prism is greater than the altitude. (?)

10. A **regular prism** is a right prism whose bases are regular polygons.

11. Prisms are *triangular*, *quadrangular*, *pentangular*, ... according as their bases are *triangles*, *quadrilaterals*, *pentagons*, ...

12. A **truncated prism** is the portion of a prism included between either base and a section cutting all the lateral edges.

13. A **right section** of a prism is a section perpendicular to its lateral edges.

14. A **parallelopiped** is a prism whose bases are parallelograms.

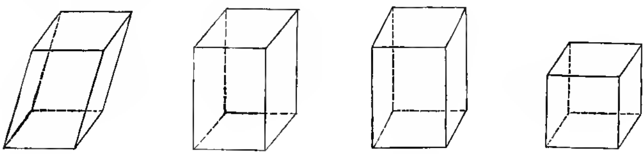
15. An **oblique parallelopiped** is a parallelopiped whose lateral edges are oblique to its bases.

16. A **right parallelopiped** is a parallelopiped whose lateral edges are perpendicular to its bases. Hence, its lateral faces are rectangles. (?)

17. A **rectangular parallelopiped** is a right parallelopiped whose bases are rectangles.

18. The **dimensions** of a rectangular parallelopiped are three edges having a common vertex.

19. A **cube** is a rectangular parallelopiped all of whose faces are squares.

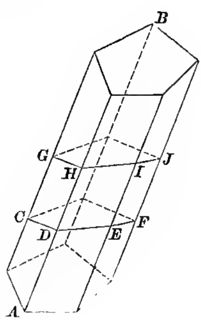


20. A cube whose edge is a linear unit is usually taken as the unit of volume.

377. Proposition I.—Theorem.

Parallel sections of a prism are equal polygons.

Let CF, GJ, be parallel sections of the prism AB; then, CF and GJ are equal.



For, CD and GH are parallel (?) and equal; (?) also, DE and HI, . . .

$$\therefore \quad CDE = GHI, \ldots \quad (?)$$

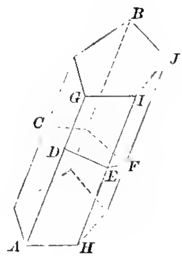
Hence, the sections are mutually equilateral and equiangular, and therefore equal.

378. Corollary.

Any section of a prism parallel to the base is equal to the base. (?)

379. Proposition II.—Theorem.

The lateral area of a prism is equal to the product of a lateral edge by the perimeter of a right section.



Let CF be a right section of the prism AB. The lateral edges are all equal. (?) Let l denote the lateral area; e, a lateral edge; p, the perimeter of the right section.

The face AI is a parallelogram, whose base is AG and altitude DE.

$$\therefore \quad AI = AG \times DE;$$

also, $\quad HJ = AG \times EF, \ldots$

$$\therefore \quad AI + HJ + \ldots = AG \times (DE + EF + \ldots),$$

$$\therefore \quad l = e \times p.$$

Remark.—Let b denote the area of each base, and s the entire surface. Then, $s = e \times p + 2b$.

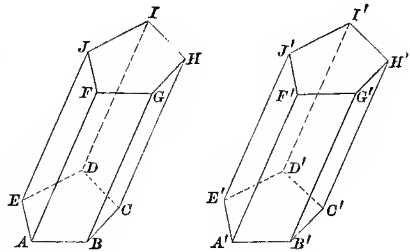
380. Corollary.

The lateral area of a right prism is equal to the perimeter of the base multiplied by the altitude. (?)

381. Proposition III.—Theorem.

Two prisms are equal if the three faces including a trihedral angle of the one are respectively equal to the three corresponding faces including a trihedral angle of the other.

Let the three faces, AD, AG, AJ, including the trihedral angle A of the prism AI, be respectively equal to the three faces, $A'D'$, $A'G'$, $A'J'$, including the trihedral angle A' of the prism $A'I'$, and similarly placed.



The trihedral angles, A, A', are equal, (?) and can be made to coincide; then, the base AD will coincide with $A'D'$, the face AG with $A'G'$, and AJ with $A'J'$; hence, FG will coincide with $F'G'$, FJ with $F'J'$. Therefore, the planes of the upper bases coincide, (?) also, the planes of the faces, BH, $B'H'$; hence, GH, $G'H'$, coincide, and therefore the faces, BH, $B'H'$.

In like manner, it can be shown that the remaining faces respectively coincide; hence, the prisms coincide and are therefore equal.

382. Corollaries.

1. *Two truncated prisms are equal if the three faces including a trihedral angle of the one are respectively equal to the three faces including a trihedral angle of the other, and are similarly placed.*

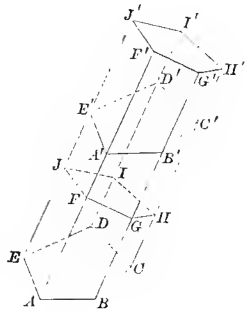
The proof is the same as for the last proposition.

2. *Two right prisms having equal bases and equal altitudes are equal.*

If the faces are not similarly placed, the prisms can be made to coincide if one be inverted.

383. Proposition IV.—Theorem.

Any oblique prism is equivalent to a right prism whose bases are equal to right sections of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.

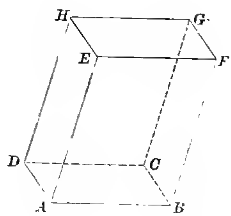


Let AD' be an oblique prism; FI, a right section. Complete the right prism FI', making its edges equal to those of the oblique prism.

The truncated prisms, AI, $A'I'$, are equal. (?) To each add the truncated prism FD'. Then the oblique prism AD' is equivalent to the right prism FI'.

384. Proposition V.—Theorem.

Any two opposite faces of a parallelopiped are equal and parallel.



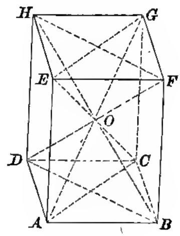
Let AG be a parallelopiped. The bases, AC, EG, are equal and parallel. (?) AE, DH, are equal and parallel, (?) so, also, are AB, DC; hence the angles, EAB, HDC, are equal, and their planes parallel. (?) Therefore the opposite faces, AF, DG, are equal (**83**, 2) and parallel.

In like manner, it can be proved that AH and BG are equal and parallel.

Any two opposite faces of a parallelopiped can be taken for bases, since they are equal and parallel parallelograms.

385. Proposition VI.—Theorem.

The four diagonals of a parallelopiped bisect each other.



Any two diagonals are the diagonals of a parallelogram, (?) and hence bisect each other. Any three diagonals, therefore, pass through the middle point of the fourth; hence, all the diagonals pass through the same point, which is called the center.

386. Proposition VII.—Theorem.

The sum of the squares of the four diagonals of a parallelopiped is equivalent to the sum of the squares of the twelve edges.

From the diagram of the last proposition we have

$$\overline{AG}^2 + \overline{CE}^2 = \overline{AE}^2 + \overline{CG}^2 + \overline{AC}^2 + \overline{EG}^2, \text{ (?)}$$

$$\overline{BH}^2 + \overline{DF}^2 = \overline{BF}^2 + \overline{DH}^2 + \overline{BD}^2 + \overline{FH}^2.$$

Adding, substituting $\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2$ for $\overline{AC}^2 + \overline{BD}^2$, and $\overline{EF}^2 + \overline{FG}^2 + \overline{GH}^2 + \overline{HE}^2$ for $\overline{EG}^2 + \overline{FH}^2$, we have $\overline{AG}^2 + \overline{CE}^2 + \overline{BH}^2 + \overline{DF}^2$ $= \overline{AE}^2 + \overline{BF}^2 + \overline{CG}^2 + \overline{DH}^2 + \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2$ $+ \overline{DA}^2 + \overline{EF}^2 + \overline{FG}^2 + \overline{GH}^2 + \overline{HE}^2.$

387. Corollary.

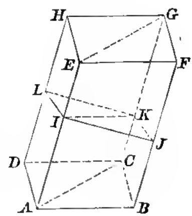
The square of a diagonal of a rectangular parallelopiped is equivalent to the sum of the squares of three edges meeting at a common vertex. (?)

REMARK.—A parallelopiped can be constructed upon any three given straight lines not in the same plane, terminating in the same point, by passing a plane through the extremity of each parallel to the plane of the other two. These planes, together with the planes of the given lines, will by their mutual intersections determine the faces of the parallelopiped.

388. Proposition VIII.—Theorem.

The plane embracing two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.

The plane AG, embracing the diagonally opposite edges, AE, CG, divides the parallelopiped DF into two equivalent triangular prisms ABC-F, ADC-H.



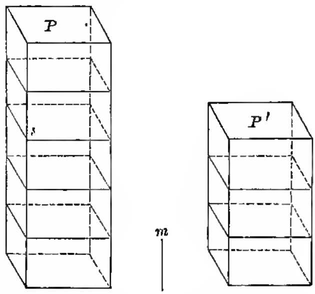
For, the right section LJ is a parallelogram. (?) IK, the intersection of LJ with the plane AG, is the diagonal of the parallelogram LJ, and hence divides it into two equal triangles, IJK, ILK.

The prism ABC-F is equivalent to the right prism whose base is IJK and whose altitude is AE. (?) The prism ADC-H is equivalent to the right prism whose base is ILK and whose altitude is AE. But the two right prisms are equal; (?) hence, the prisms, ABC-F, ADC-H, are equivalent.

389. Proposition IX.—Theorem.

Two rectangular parallelopipeds having equal bases are proportional to their altitudes.

Let P, P', be two rectangular parallelopipeds having equal bases, and let a, a', be their altitudes.



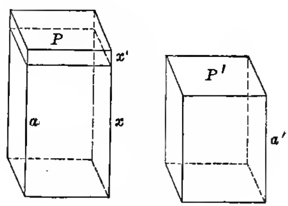
1. If a, a', are commensurable, let their common measure m be contained, for example, 5 times in a and 3 times in a'.

$$\therefore \quad a : a' :: 5 : 3.$$

If sections parallel to the bases divide the altitudes, a, a', into 5 and 3 equal parts respectively, P will be divided into 5 equal parallelopipeds, and P' into 3 equal parallelopipeds, each equal to each of those in P. (?)

$$\therefore \quad P : P' :: 5 : 3.$$

$$\therefore \quad P : P' :: a : a'.$$



2. If a, a', are incommensurable, divide a' into any number of equal parts, and apply one part to a as many times as possible.

The section through the last point of division of a divides P into two parallelopipeds, Q, Q', whose altitudes are x, x', the altitude x being commensurable with a', and x' being less than one of the equal divisions of a'.

$$\therefore \quad \frac{Q}{P'} = \frac{x}{a'}. \quad (?)$$

Now, if the number of parts into which a' is divided be

indefinitely increased, each part will indefinitely diminish, x will approach a as its limit, x' will approach 0, and Q will approach P. Hence, the limit of $\frac{Q}{P'}$ is $\frac{P}{P'}$, and the limit of $\frac{x}{a'}$ is $\frac{a}{a'}$. (?) $\therefore$ $\frac{P}{P'} = \frac{a}{a'}$. (?)

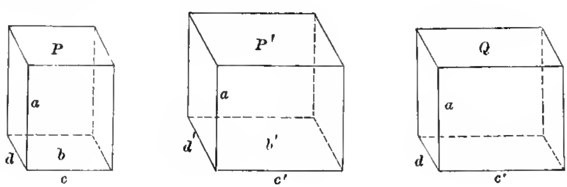
390. Corollary.

Two rectangular parallelopipeds having two dimensions of one respectively equal to two dimensions of the other are proportional to their third dimensions. (?)

391. Proposition X.—Theorem.

Two rectangular parallelopipeds having equal altitudes are proportional to their bases.

Let P, P', be two rectangular parallelopipeds whose dimensions are a, c, d, and a, c', d', respectively, and whose bases are b, b'.



Let Q be a rectangular parallelopiped having two dimensions, a, d, the same as P, and two dimensions, a, c', the same as P'. Then,

$$P : Q :: c : c', \text{ and } Q : P' :: d : d'. \text{ (?)}$$

Taking the product of the corresponding terms of these proportions, and omitting the common factor Q from the first couplet,

$$P : P' :: c \times d : c' \times d'.$$

But, $\qquad b = c \times d$, and $b' = c' \times d'$. (?)

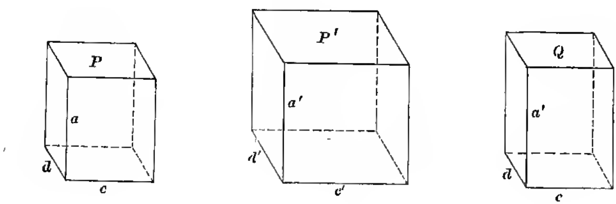
$\therefore \qquad P : P' :: b : b'.$

392. Corollary.

Two rectangular parallelopipeds having a dimension of one equal to a dimension of the other are proportional to the products of their other two dimensions. (?)

393. Proposition XI.—Theorem.

Any two rectangular parallelopipeds are proportional to the products of their three dimensions.



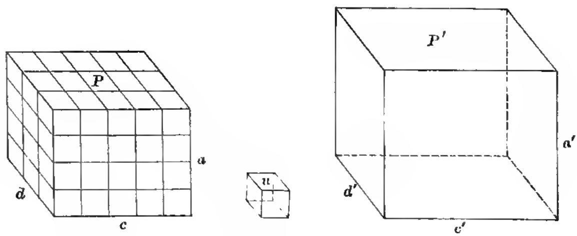
Let P, P', be two rectangular parallelopipeds whose dimensions are a, c, d, and a', c', d', respectively, and Q a rectangular parallelopiped having two dimensions, c, d, the same as P, and the third dimension, a', the same as P'.

Then, $\quad P : Q :: a : a'$, and $Q : P' :: c \times d : c' \times d'$. (?)

$\therefore \qquad P : P' :: a \times c \times d : a' \times c' \times d'.$

394. Proposition XII.—Theorem.

The volume of a rectangular parallelopiped is equal to the product of its three dimensions.



1. If the dimensions of the parallelopiped are commensurable, let P be its volume; a, c, d, its dimensions; and the cube, whose edge is a common measure of a, c, d, the unit of volume.

The edge of this cube, taken as the linear unit, is contained in the dimensions, respectively, a, c, d times.

Sections parallel to the faces, dividing the edges into parts each equal to the linear unit, divide the parallelopiped into cubes each equal to u, the unit of volume. These cubes are arranged in a strata, each containing d rows of c cubes each, and hence, $c \times d$ cubes; therefore, a strata contain $a \times c \times d$ cubes, or units of volume.

$$\therefore \quad P = a \times c \times d.$$

2. If the dimensions of the parallelopiped are incommensurable, let P' be its volume, a', c', d', its dimensions.

Then, $\quad P : P' :: a \times c \times d : a' \times c' \times d'$. (?)

But, $\qquad P = a \times c \times d; \quad \therefore \quad P' = a' \times c' \times d'$.

395. Corollaries.

1. *The volume of a rectangular parallelopiped is equal to the product of its base by its altitude.*

For, let b denote the base; then,

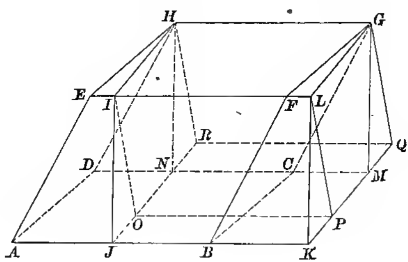
$$b = c \times d; \quad \therefore \quad P = b \times a.$$

2. *The volume of a cube is equal to the third power of its edge.*

For, if $d = c = a$, $P = a \times a \times a = a^3$.

396. Proposition XIII.—Theorem.

The volume of any parallelopiped is equal to the product of its base by its altitude.



Let P denote the volume; b, the base $ABCD$; a, the altitude HR of the parallelopiped $ABCD$-F, having all its faces oblique.

The oblique parallelopiped $ABCD$-F is equivalent to the right parallelopiped $HIJN$-L, whose base is the right section $HIJN$ of the oblique parallelopiped, and whose altitude HG is a lateral edge of the oblique parallelopiped. (?)

The right parallelopiped $HIJN$-L, regarded as an oblique parallelopiped whose base is $IJKL$ and lateral edge IH, is equivalent to the right parallelopiped $IOPL$-R, whose base is the right section $IOPL$ and whose altitude is the lateral edge IH.

But this right parallelopiped is a rectangular parallelo-

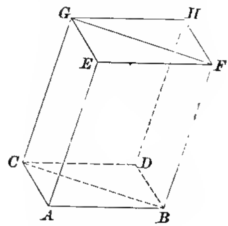
piped, (?) whose base $OPQR = ILGH = EFGH = ABCD = b$, and whose altitude $HR = a$.

Now, $ABCD\text{-}F = JKMN\text{-}L = OPQR\text{-}L = b \times a$.

$\therefore \qquad P = b \times a.$

397. Proposition XIV.—Theorem.

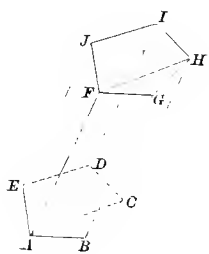
The volume of any prism is equal to the product of its base by its altitude.



1. If the base is a triangle, let P denote the volume, b the base, a the altitude.

The parallelopiped AH, constructed on the edges, AB, AC, AE, of the triangular prism $ABC\text{-}E$, has $2P$ for its volume, $2b$ for its base, and a for its altitude. (?)

$\therefore \quad 2P = 2b \times a$; (?) $\therefore \quad P = b \times a$.



2. If the base is a polygon having more than three sides, let P denote the volume, b the base, a the altitude.

Planes embracing a common lateral edge and its opposite lateral edges respectively, divide the prism into triangular prisms having a for their common altitude. Denoting the volumes of these triangular prisms by P', P'', ... and their bases by b', b'', ... we have

$$P' + P'' + \ldots = b' \times a + b'' \times a + \ldots = (b' + b'' + \ldots) \times a.$$

But, $\quad P' + P'' + \ldots = P, \qquad b' + b'' + \ldots = b.$

$\therefore \qquad P = b \times a.$

398. Corollaries.

1. *Any two prisms are proportional to the products of their bases by their altitudes.*

Let P, P', denote the volumes; b, b', the bases; a, a', the altitudes of any two prisms. Then,

$$P = b \times a, \quad P' = b' \times a'; \quad \therefore \quad \frac{P}{P'} = \frac{b \times a}{b' \times a'}.$$

2. *Two prisms having equivalent bases are proportional to their altitudes.*

For, if $b' = b$, $\dfrac{P}{P'} = \dfrac{b \times a}{b \times a'}$; $\therefore$ $\dfrac{P}{P'} = \dfrac{a}{a'}$.

3. *Two prisms having equal altitudes are proportional to their bases.*

For, if $a' = a$, $\dfrac{P}{P'} = \dfrac{b \times a}{b' \times a}$; $\therefore$ $\dfrac{P}{P'} = \dfrac{b}{b'}$.

399. Exercises.

1. The three edges of a rectangular parallelopiped are l, m, n; find its diagonal, surface, and volume.

2. The surface of a cube is s; find its edge, diagonal, and volume.

3. The volume of a triangular prism is equal to the product of the area of a lateral face by one-half the perpendicular from the opposite edge to that face.

4. Of all quadrangular prisms having equivalent surfaces, the cube has the maximum volume.

II. PYRAMIDS.

400. Definitions.

1. A **pyramid** is a polyhedron one of whose faces is a polygon, and whose other faces are triangles having a common vertex and the sides of the polygon for bases.

2. The **base** of a pyramid is the face whose sides are the bases of the triangles having a common vertex.

3. The **lateral faces** of a pyramid are all the faces except the base.

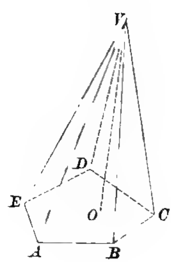
4. The **lateral surface** of a pyramid is the sum of its lateral faces.

5. The **lateral edges** of a pyramid are the intersections of its lateral faces.

6. The **basal edges** of a pyramid are the intersections of its base with its lateral faces.

7. The **vertex** of a pyramid is the common vertex of its lateral faces.

8. The **altitude** of a pyramid is the perpendicular from its vertex to the plane of its base.



Thus, V-$ABCDE$ is a pyramid; $ABCDE$ is its base; AVB, BVC, ... are its lateral faces, whose sum is its lateral surface; AV, BV, ... are its lateral edges; AB, BC, ... its basal edges; V is its vertex; VO, its altitude.

9. A **regular pyramid** is a pyramid whose base is a regular polygon, and whose vertex is in the perpendicular to the base at its center.

10. The **axis** of a regular pyramid is the straight line joining its vertex with the center of its base.

11. The **slant height** of a regular pyramid is the altitude of any lateral face.

12. A pyramid is *triangular, quadrangular, pentangular,* . . . according as its base is a *triangle, quadrilateral, pentagon,* . . .

13. A **truncated pyramid** is the portion of a pyramid included between its base and a section cutting all its lateral edges.

14. A **frustum** of a pyramid is a truncated pyramid in which the cutting section is parallel to its base.

15. The base of the pyramid is called the *lower base* of the frustum, and the parallel section, its *upper base.*

16. The altitude of a frustum is the perpendicular from either base to the plane of the other.

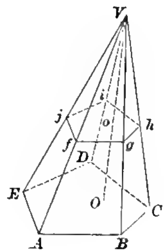
17. The lateral faces of a frustum of a regular pyramid are the trapezoids included between its bases; the lateral surface is the sum of the lateral faces.

18. The slant height of a frustum of a regular pyramid is the altitude of any lateral face.

401. Proposition XV.—Theorem.

Any section of a pyramid parallel to its base divides the edges and altitude proportionally, and is similar to the base.

The section *fghij* of the pyramid *V-ABCDE*, parallel to the base, divides the edges, *VA*, *VB*, .. and the altitude *VO* proportionally.



For, conceive a plane through *V* parallel to the base; then the edges and altitude will be intersected by three parallel planes. Therefore,

$$Vf : VA :: Vg : VB \ldots :: Vo : VO.$$

The section *fghij* and the base *ABCDE* are similar polygons.

For, *fg*, *AB*, are parallel; also, *gh*, *BC*...; hence, the angles, *jfg*, *EAB*, are equal; also, *fgh*, *ABC*...

The triangles, *Vfg*, *VAB*, are similar; (?) also, *Vgh*, *VBC*...

$$\therefore \quad \frac{fg}{AB} = \frac{Vg}{VB} = \frac{gh}{BC} = \frac{Vh}{VC} = \frac{hi}{CD} = \ldots$$

$$\therefore \quad fg : AB :: gh : BC :: hi : CD :: \ldots$$

Therefore, *fghij*, *ABCDE*, are similar. (?)

402. Corollaries.

1. *Any section of a pyramid parallel to its base is to the base as the square of the perpendicular from the vertex to the plane of the section is to the square of the altitude of the pyramid.*

For, $fghij : ABCDE :: \overline{fg}^2 : \overline{AB}^2 :: \overline{Vf}^2 : \overline{VA}^2 :: \overline{Vo}^2 : \overline{VO}^2.$

2. *If two pyramids have equal altitudes, sections parallel to the bases and whose planes are at equal distances from the vertices are proportional to the bases.*

For, let *b*, *b′*, denote the bases of two pyramids; *a*, their

common altitude; s, s', sections parallel to the bases and whose planes are at the common distance d from the vertices.

Then, $s : b :: d^2 : a^2$, and $s' : b' :: d^2 : a^2$.

$\therefore \quad s : b :: s' : b'$, $\quad \therefore \quad s : s' :: b : b'$.

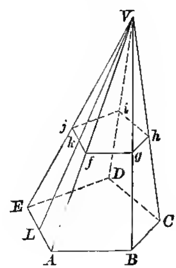
3. *If two pyramids have equal altitudes and equivalent bases, sections parallel to the bases and whose planes are at equal distances from the vertices are equivalent.*

For, since $s : s' :: b : b'$, if $b = b'$, $s = s'$.

403. Proposition XVI.—Theorem.

The lateral area of a regular pyramid is equal to one-half the product of the perimeter of its base by its slant height.

Let l denote the lateral surface of the regular pyramid V-$ABCDE$; e, a basal edge; p, the perimeter of the base; t, one of the equal (?) lateral faces; n, the number of lateral faces; h, the slant height. Then,



$t = \frac{1}{2}e \times h$; $\quad \therefore \quad nt = \frac{1}{2}ne \times h$.

But, $l = nt$, and $p = ne$;

$\therefore \quad l = \frac{1}{2}p \times h$.

REMARK.—Let b denote the area of the base, and s the entire surface. Then, $s = \frac{1}{2}p \times h + b$.

404. Corollary.

The lateral area of a frustum of a regular pyramid is equal to one-half the product of the sum of the perimeters of its bases by its slant height.

For, let l denote the lateral surface of the frustum; e', an edge of the upper base; p', its perimeter; t, one of the equal (?) lateral faces; h, the slant height; e, n, p, the same as above. Then,

$$t = \tfrac{1}{2}(e + e') \times h; \quad \therefore \quad nt = \tfrac{1}{2}(ne + ne') \times h.$$

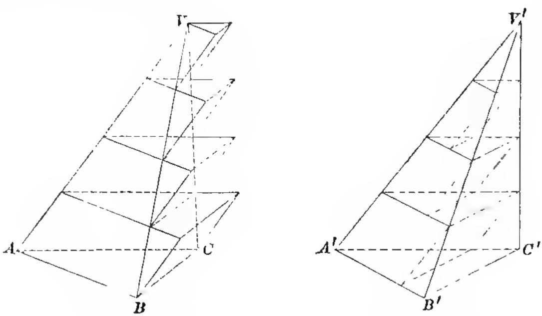
But, $l = nt$, and $p = ne$, $p' = ne'$;

$$\therefore \quad l = \tfrac{1}{2}(p + p') \times h.$$

REMARK.—Let b, b', denote the areas of the bases, and s the entire surface. Then, $s = \frac{1}{2}(p + p') \times h + b + b'$.

405. Proposition XVII.—Theorem.

Two triangular pyramids having equivalent bases and equal altitudes are equivalent.



Let P, P', denote the volumes of two triangular pyramids, V-ABC, V'-$A'B'C'$, having equivalent bases and equal altitudes; then, $P = P'$.

If the pyramids are not equivalent, let $P > P'$.

Corresponding sections of the two pyramids, parallel to the bases, dividing the altitudes into the same number of equal parts, are equivalent. (?)

On the lower base and the several sections of the pyramid V-ABC, as lower bases, construct prisms whose lateral edges are parallel to the lateral edge VA, and whose altitudes are the equal divisions of the altitude of the pyramid.

On the sections of the pyramid V'-$A'B'C'$, as upper bases, construct prisms whose lateral edges are parallel to the lateral edge $V'A'$, and whose altitudes are the equal divisions of the altitude of the pyramid.

Let S denote the sum of the volumes of the first set of prisms, S' that of the second set.

Now, $P < S$, and $P' > S'$; $\therefore$ $P - P' < S - S'$.

Except the lower prism of the first set, each prism of either set has an equivalent prism in the other set, since the two have equivalent bases and equal altitudes.

Let p denote the volume of the lower prism of the first set. Then,

$$S - S' = p; \quad \therefore \quad P - P' < p.$$

By indefinitely increasing the number of equal parts of the altitudes of the pyramids, the altitudes of the prisms constructed as above will be indefinitely diminished; hence, taking l as the symbol for limit,

$$lp = 0; \ (?) \quad \therefore \quad P - P' = 0; \quad \therefore \quad P = P'.$$

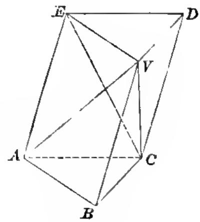
406. Proposition XVIII.—Theorem.

The volume of any pyramid is equal to one-third of the product of its base by its altitude.

1. If V-ABC is a triangular pyramid, let P denote its volume; b, its base; a, its altitude.

On the base ABC construct a prism ABC-E whose lateral

edges are parallel to the edge BV, and whose altitude is equal to that of the pyramid.

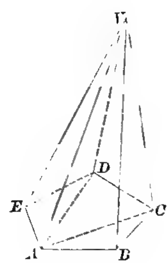


The face AVC of the pyramid divides the prism into two pyramids, one of which is the given pyramid $V\text{-}ABC$, and the other the quadrangular pyramid $V\text{-}ACDE$.

The plane VCE divides the quadrangular pyramid into the triangular pyramids, $V\text{-}CDE$, $V\text{-}ACE$, which are equivalent, since they have equal bases (?) and the same altitude. (?)

The pyramid $V\text{-}CDE$ is the same as the pyramid $C\text{-}VDE$; but $C\text{-}VDE$ and $V\text{-}ABC$ are equivalent, since they have equal bases (?) and equal altitudes. (?) Hence, the three pyramids, $V\text{-}ABC$, $C\text{-}VDE$, $V\text{-}ACE$, are equivalent: but the sum of these pyramids is equivalent to the triangular prism $ABC\text{-}E$. Therefore, the triangular pyramid $V\text{-}ABC$ is one-third of the prism $ABC\text{-}E$, having the same base and altitude. But the volume of the prism is equal to the product of its base by its altitude; hence, the volume of the pyramid is equal to one-third of the product of its base by its altitude.

$$\therefore \quad P = \tfrac{1}{3} b \times a.$$



2. If $V\text{-}ABCDE$ is a pyramid whose base is a polygon having more than three sides, let P denote its volume; b, its base; a, its altitude.

Planes embracing a common lateral edge and its opposite lateral edges respectively, divide the pyramid into triangular pyramids having a for their common altitude.

Denoting the volumes of these triangular pyramids by P', P'', ... their bases by b', b'', ... we have

$$P' + P'' + \ldots = \tfrac{1}{3} b' \times a + \tfrac{1}{3} b'' \times a + \ldots = \tfrac{1}{3}(b' + b'' + \ldots) \times a.$$

But, $\quad P' + P'' + \ldots = P, \qquad b' + b'' + \ldots = b;$

$$\therefore \qquad P = \tfrac{1}{3} b \times a.$$

407. Corollaries.

1. *Any two pyramids are proportional to the products of their bases by their altitudes.*

Let P, P', denote the volumes, b, b', the bases, a, a', the altitudes of any two pyramids. Then,

$$P = \tfrac{1}{3} b \times a, \quad P' = \tfrac{1}{3} b' \times a'; \quad \therefore \quad \frac{P}{P'} = \frac{b \times a}{b' \times a'}.$$

2. *Two pyramids having equivalent bases are proportional to their altitudes.* (?)

3. *Two pyramids having equal altitudes are proportional to their bases.* (?)

4. *The volume of a frustum of any pyramid is equal to one-third of the product of its altitude by the sum of its lower base, its upper base, and the mean proportional between its bases.*

Let F denote the volume of the frustum; b, b', its bases; a', a, $a' - a$, the altitudes of the pyramid, the frustum, and the part above the frustum.

Then, $\quad F = \tfrac{1}{3} a'b - \tfrac{1}{3}(a' - a)\, b'.$

$\therefore \quad (1) \quad F = \tfrac{1}{3}[a'(b - b') + ab'].$

But, $\quad b : b' :: a'^2 : (a' - a)^2 \quad$ **(402, 1)**.

$\therefore \quad \sqrt{b} : \sqrt{b'} :: a' : a' - a. \quad$ (?)

$\therefore \quad a'\sqrt{b} - a\sqrt{b} = a'\sqrt{b'}.$

$\therefore \quad a'(\sqrt{b} - \sqrt{b'}) = a\sqrt{b}.$

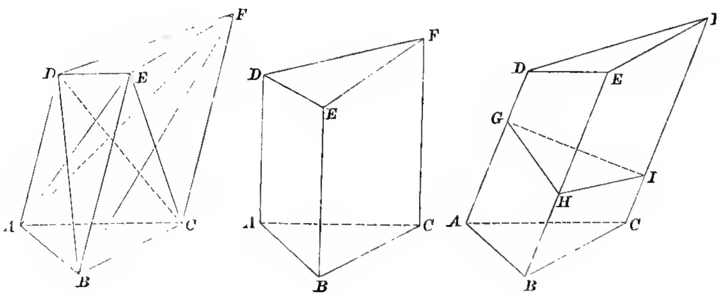
$$\therefore \quad a'(\sqrt{b}-\sqrt{b'})(\sqrt{b}+\sqrt{b'}) = a\sqrt{b}(\sqrt{b}+\sqrt{b'}).$$

$$\therefore \qquad (2) \quad a'(b-b') = a\,(b+\sqrt{bb'}).$$

Substituting (2) in (1), $\qquad F = \frac{1}{3}a\,(b+b'+\sqrt{bb'}).$

408. Proposition XIX.—Theorem.

The volume of any truncated triangular prism is equal to one-third of the product of its right section by the sum of its lateral edges.



Let the truncated triangular prism *ABC-D* be divided by the planes, *AEC*, *DEC*, into three pyramids, *E-ABC*, *E-ACD*, *E-CDF*.

The pyramid *E-ABC* has *ABC* for its base and *E* for its vertex. The pyramid *E-ACD* is equivalent to the pyramid *B-ACD*, having the same base, *ACD*, and an equal altitude. (?) But *B-ACD* is the same as *D-ABC*; hence, *E-ACD* is equivalent to *D-ABC*, which has *ABC* for its base and *D* for its vertex.

The pyramid *E-CDF* is equivalent to the pyramid *B-ACF*, having an equivalent base (?) and an equal altitude. (?) But *B-ACF* is the same as *F-ABC*; hence, *E-CDF* is equivalent to *F-ABC*, which has *ABC* for its base and *F* for its vertex.

Hence, the truncated prism *ABC-D* is equivalent to three pyramids, *E-ABC, D-ABC, F-ABC,* whose common base is *ABC* and whose vertices are *E, D, F.*

If the truncated prism is right, as in the second figure, its lateral edges are respectively the altitudes of the three pyramids.

$$\therefore \quad ABC\text{-}D = \tfrac{1}{3}ABC \times BE + \tfrac{1}{3}ABC \times AD + \tfrac{1}{3}ABC \times CF.$$

$$\therefore \quad ABC\text{-}D = \tfrac{1}{3}ABC \times (BE + AD + CF).$$

The right section *GHI* of the truncated prism *ABC-D*, in the third figure, divides *ABC-D* into two right truncated prisms, *GHI-A, GHI-D,* having a common base *GHI.*

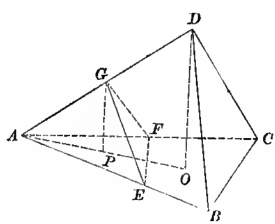
$$\therefore \quad ABC\text{-}D = \tfrac{1}{3}GHI \times (BH + AG + CI) + \tfrac{1}{3}GHI \times (HE + GD + IF).$$

$$\therefore \quad ABC\text{-}D = \tfrac{1}{3}GHI \times (BE + AD + CF).$$

409. Proposition XX.—Theorem.

Two tetrahedrons having a trihedral angle of one equal to a trihedral angle of the other are proportional to the products of the three edges of these trihedral angles.

The tetrahedrons, *D-ABC, G-AEF,* having coincident trihedral angles at *A,* have *ABC, AEF,* for their bases, and *DO, GP,* perpendiculars from the vertices, *D, G,* to the plane of the bases, for their altitudes.



$$\therefore \quad \frac{D\text{-}ABC}{G\text{-}AEF} = \frac{ABC \times DO}{AEF \times GP} = \frac{ABC}{AEF} \times \frac{DO}{GP}. \quad (?)$$

But, $\frac{ABC}{AEF} = \frac{AB \times AC}{AE \times AF}$, (?) $\qquad \frac{DO}{GP} = \frac{AD}{AG}$. (?)

$$\therefore \quad \frac{D\text{-}ABC}{G\text{-}AEF} = \frac{AB \times AC \times AD}{AE \times AF \times AG}.$$

410. Exercises.

1. The lateral surface of a pyramid is greater than the base.

2. The planes embracing the three edges of any trihedral angle of a tetrahedron and the medial lines of the opposite face intersect in a straight line.

3. The four lines joining the vertices of a tetrahedron with the intersections of the medial lines of the opposite faces, meet in a point which divides each of these lines in the ratio of 1 to 4.

4. The straight lines joining the middle points of the opposite edges of any tetrahedron meet in a point which bisects each of these lines.

5. The plane bisecting a dihedral angle of a tetrahedron divides the opposite edge into segments proportional to the adjacent faces.

6. A plane embracing any edge and the middle point of the opposite edge of a tetrahedron divides it into two equivalent solids.

7. The volume of a truncated triangular prism is equal to the product of its lower base by the perpendicular to the lower base from the intersection of the medial lines of the upper base.

III. SIMILAR POLYHEDRONS.

411. Definitions.

1. **Similar polyhedrons,** being identical in form, are polyhedrons having the same number of faces, respectively similar and similarly placed, and their corresponding polyhedral angles equal.

2. **Homologous faces, lines,** or **angles** of similar polyhedrons are faces, lines, or angles similarly placed.

412. Proposition XXI.—Theorem.

The ratio of similitude of any two homologous faces of two similar polyhedrons is equal to the ratio of similitude of any other two homologous faces.

For, the ratio of similitude of any two homologous faces is the ratio of any two homologous sides of these faces; and, since these sides are homologous sides of adjacent faces, the ratio of similitude of any two homologous faces is equal to that of any two adjacent homologous faces; and so on for all the faces.

The ratio of similitude of any two homologous faces of two similar polyhedrons is called the *ratio of similitude of the polyhedrons.*

413. Corollaries.

1. *The ratio of similitude of two similar polyhedrons is equal to the ratio of any two homologous edges.* (?)

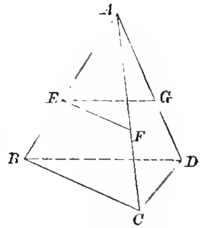
2. *Any two homologous faces of two similar polyhedrons are proportional to the squares of any two homologous edges.* (?)

3. *The surfaces of two similar polyhedrons are proportional to the squares of any two homologous edges.* (?)

414. Proposition XXII.—Theorem.

The portion of a tetrahedron cut off by a section parallel to any face is a tetrahedron similar to the given tetrahedron.

Let $ABCD$ be a tetrahedron; EFG, a section parallel to BCD. Then, $AEFG$ is a tetrahedron similar to $ABCD$.



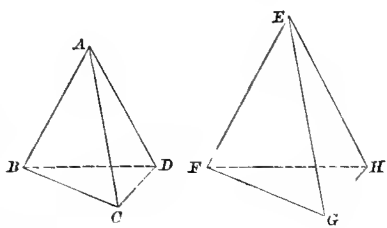
EFG divides the edges, AB, AC, AD, proportionally; (?) hence, the faces, AEF, ABC, are similar; also, AFG, ACD, and AGE, ADB; (?) and EFG, BCD, are also similar. (?)

The corresponding trihedral angles are equal. (?) Therefore, $AEFG$ is a tetrahedron similar to $ABCD$.

415. Proposition XXIII.—Theorem.

Two tetrahedrons having a dihedral angle of one equal to a dihedral angle of the other, and the faces including these angles respectively similar and similarly placed, are similar.

Let the tetrahedrons, $ABCD$, $EFGH$, have the dihedral angles, AB, EF, equal, and the faces, ABC, ABD, respectively similar to EFG, EFH; then, these tetrahedrons are similar.



The trihedral angles, A, E, can be made to coincide in all their parts, (?) and hence are equal. Therefore the angles, CAD, GEH, are equal.

From the given similar faces we have the proportions,

$$\left.\begin{array}{l} AC : EG :: AB : EF. \\ AD : EH :: AB : EF. \end{array}\right\} \quad \therefore \quad AC : EG :: AD : EH.$$

Therefore the faces, *CAD*, *GEH*, are similar. (?)

In like manner, it can be proved that the trihedral angles, *B*, *F*, are equal, and that the faces, *BCD*, *FGH*, are similar.

The trihedral angles, *C*, *G*, are equal, since their facial angles are respectively equal (?) and similarly placed; the trihedral angles, *D*, *H*, are equal. (?) Therefore the tetrahedrons are similar.

416. Corollaries.

1. *Two similar polyhedrons can be decomposed into the same number of tetrahedrons, respectively similar and similarly placed.*

Select two homologous vertices; divide the homologous faces not adjacent to these vertices into the same number of triangles, respectively similar and similarly placed; draw lines from the selected vertices to the vertices of these triangles. The planes of these lines will divide the polyhedrons as stated. (?)

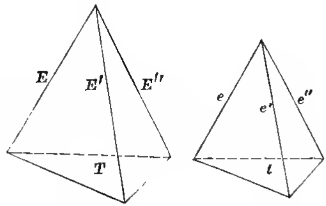
2. *Two polyhedrons composed of the same number of tetrahedrons, respectively similar and similarly placed, are similar.* (?)

3. *The ratio of any two homologous lines of two similar polyhedrons is equal to the ratio of any two homologous edges, and therefore equal to the ratio of similitude of the polyhedrons.* (?)

4. *In the corollaries of* **413,** *lines may be substituted for edges.*

417. Proposition XXIV.—Theorem.

Similar polyhedrons are proportional to the cubes of any two homologous lines.



1. Let T, t, denote the volumes of two similar tetrahedrons; E, E', E'', and e, e', e'', respectively, homologous edges of two homologous trihedral angles; L, l, any two homologous lines. Then,

$$\frac{T}{t}=\frac{E\times E'\times E''}{e\times e'\times e''}=\frac{E}{e}\times\frac{E'}{e'}\times\frac{E''}{e''}; \text{ but, } \frac{E}{e}=\frac{E'}{e'}=\frac{E''}{e''}.$$

$$\therefore\quad \frac{T}{t}=\frac{E}{e}\times\frac{E}{e}\times\frac{E}{e}=\frac{E^3}{e^3}, \text{ or } T : t :: E^3 : e^3.$$

But, $\quad E : e :: L : l, \quad \therefore\quad E^3 : e^3 :: L^3 : l^3.$

$\therefore\quad T : t :: L^3 : l^3.$

2. Let T, T', ... t, t', ... respectively, denote the similar tetrahedrons into which any two similar polyhedrons, P, p, can be divided: L, L', ... l, l', ... respectively, homologous lines of the similar tetrahedrons.

Then, $\left\{\begin{array}{l} T : t :: L^3 : l^3. \\ T' : t' :: L'^3 : l'^3. \\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\}$ But, $\left\{\begin{array}{l} \quad L : l :: L' : l' \ldots \\ \therefore\ L^3 : l^3 :: L'^3 : l'^3 \ldots \end{array}\right.$

$\therefore\quad T : t :: T' : t' \ldots\ldots\ldots :: L^3 : l^3.$

$\therefore\quad T+T'+\ldots : t+t'+\ldots :: L^3 : l^3.$

But, $\left\{\begin{array}{l} T+T'+\ldots=P. \\ t+t'+\ldots=p. \end{array}\right\} \therefore\ P : p :: L^3 : l^3.$

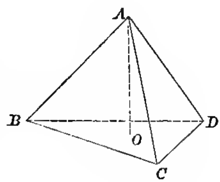
IV. REGULAR POLYHEDRONS.

418. Definition.

A **regular polyhedron** is a polyhedron all of whose faces are equal regular polygons, and all of whose polyhedral angles are equal.

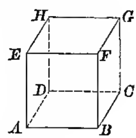
419. Constructions.

1. *To construct a regular tetrahedron.*



Construct the equilateral triangle BCD; at its center O erect the perpendicular OA, on which take the point A, whose distance from each of the vertices, B, C, D, is equal to a side of the triangle, and draw the straight lines, AB, AC, AD.

The solid $ABCD$, thus determined, is a regular tetrahedron. (?)

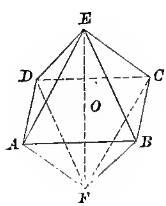


2. *To construct a regular hexahedron.*

Construct a square $ABCD$, and an equal square on each side perpendicular to the first. The upper sides of these squares will be the sides of an equal square.

The solid $ABCDF$, thus determined, is a regular hexahedron.

3. *To construct a regular octahedron.*

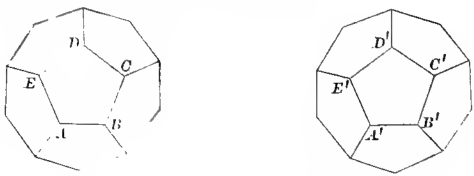


Construct a square $ABCD$; through its center O pass a perpendicular EF to its plane; from two points, E, F, on this perpendicular — one above and the other below the plane AC, whose

distances from each of the vertices, A, B, C, D, are equal to a side of the square—draw the straight lines, EA, EB, ... FA, FB, ...

The solid $EABCDF$, thus determined, is a regular octahedron. (?)

4. *To construct a regular dodecahedron.*



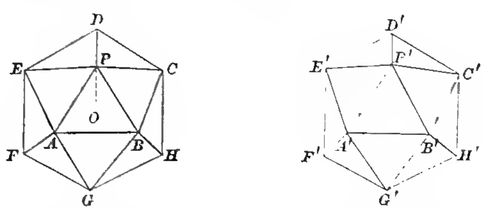
Construct a regular pentagon $ABCDE$, on each side of which construct an equal pentagon so inclined that trihedral angles shall be formed at A, B, C, D, E. The convex surface thus formed is composed of six regular pentagons.

In like manner, upon an equal pentagon $A'B'C'D'E'$ construct an equal convex surface.

Apply one of these surfaces to the other, with their convexities turned in opposite directions, so that every isolated facial angle of one shall, with two consecutive facial angles of the other, form a trihedral angle.

The solid thus determined is a regular dodecahedron. (?)

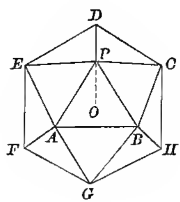
5. *To construct a regular icosahedron.*

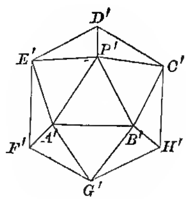


Construct a regular pentagon $ABCDE$; at its center erect a perpendicular to its plane; from a point P of this

perpendicular, whose distance from each of the vertices, A, B, C, D, E, is equal to a side of the pentagon, draw the straight lines, PA, PB, PC, PD, PE, thus forming a regular pyramid whose vertex is P.

Taking A and B as vertices, construct two pyramids, $ABPEFG$, $BAGHCP$, each equal to the first, and having two faces common with it, and two faces common with each other. There will thus be formed a convex surface composed of ten equal equilateral triangles.





In like manner, upon an equal pentagon construct an equal convex surface.

Apply one of these surfaces to the other, with their convexities turned in opposite directions, so that every combination of two facial angles of the one shall, with a combination of three facial angles of the other, form a pentahedral angle.

The solid thus determined is a regular icosahedron. (?)

420. Proposition XXV.—Theorem.

Only five regular convex polyhedrons are possible.

The faces must be equal regular polygons, and the polyhedral angles convex, each having at least three facial angles.

1. If the faces are equal equilateral triangles, convex polyhedral angles can be formed by arranging them in groups of *three*, *four*, or *five*, as in the *tetrahedron*, the *octahedron*, and the *icosahedron*.

No other combination of equilateral triangles can form a convex polyhedral angle; for, since each angle of an equilateral triangle is two-thirds of a right angle, a group of six or more would make the sum of the facial angles equal to or greater than four right angles, and therefore could not form a convex polyhedral angle, since the sum of the facial angles of a convex polyhedral angle is less than four right angles.

2. If the faces are equal squares, convex polyhedral angles can be formed by arranging them in groups of *three*, as in the regular *hexahedron* or *cube*.

A combination of four or more squares can not form a convex polyhedral angle. (?)

3. If the faces are equal regular pentagons, convex polyhedral angles can be formed by arranging them in groups of *three*, as in the regular *dodecahedron*.

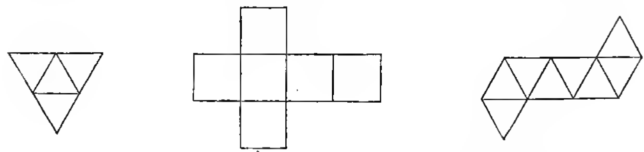
A combination of four or more regular pentagons can not form a convex polyhedral angle. (?)

4. Convex polyhedral angles can not be formed with regular polygons having more than five sides; (?) hence, polygons having more than five sides can not form the surface of regular polyhedrons, and therefore only five regular convex polyhedrons are possible.

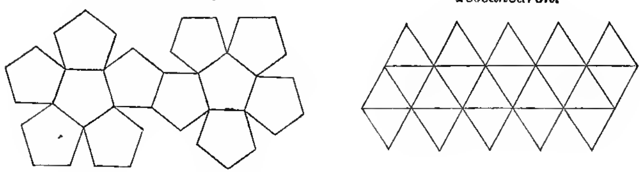
Scholium.—The regular polyhedrons can be formed thus: Draw the following diagrams on card-board; cut through

in the exterior lines and half through in the interior; the cards cut out can be folded into the regular forms.

Tetrahedron. *Hexahedron.* *Octahedron.*



Dodecahedron. *Icosahedron.*



421. Proposition XXVI.—Theorem.

1. *The number of edges of a regular polyhedron is equal to one-half the product of the number of sides of one face by the number of faces.* (?)

2. *The number of vertices of a regular polyhedron is equal to the product of the number of vertices of one face by the number of faces divided by the number of facial angles at a vertex of the polyhedron.* (?)

422. Table of Regular Polyhedrons.

NAMES.	FACES.	ANGLES.	EDGES.	VERTICES.
Tetrahedron	4 Triangles	Trihedral	6	4
Hexahedron	6 Squares	Trihedral	12	8
Octahedron	8 Triangles	Tetrahedral	12	6
Dodecahedron	12 Pentagons	Trihedral	30	20
Icosahedron	20 Triangles	Pentahedral	30	12

423. Miscellaneous Exercises.

1. Define each of the regular polyhedrons.

2. Required the positions of four equidistant points.

3. Given the edge of a regular tetrahedron, required its altitude.

4. Required the entire surface and volume of a regular hexagonal pyramid whose altitude is 24 inches, and whose base is inscribed in a circle whose diameter is 16 inches.

5. The altitude of a regular tetrahedron is equal to the sum of the four perpendiculars from any point within to the four faces.

V. SYMMETRY.—SUPPLEMENTARY.

424. Symmetry with respect to a Plane.

1. The symmetrical of a finite straight line with respect to a plane is an equal straight line. (?)

2. What is the symmetrical of an indefinite straight line with respect to a plane to which it is oblique, parallel, or perpendicular?

3. The symmetrical of a polygon with respect to a plane is an equal polygon. (?)

4. What is the symmetrical of an indefinite plane with respect to a plane to which it is oblique, parallel, or perpendicular?

5. The symmetrical of a dihedral angle with respect to a plane is an equal dihedral angle. (?)

6. Two polyhedrons symmetrical with respect to a plane have their homologous faces equal and their homologous polyhedral angles symmetrical. (?)

7. Two polyhedrons symmetrical with respect to a plane can be decomposed into the same number of tetrahedrons respectively symmetrical. (?)

8. Any two symmetrical polyhedrons are equivalent. (?)

425. Symmetry with respect to a Center.

1. The symmetrical of a polygon with respect to a center is an equal polygon. (?)

2. Two polyhedrons symmetrical with respect to a center have their homologous faces equal, and their homologous polyhedral angles symmetrical. (?)

3. The symmetrical of a polyhedron with respect to a center is equal to its symmetrical with respect to a plane. (?)

4. Two polyhedrons symmetrical with respect to a center are equivalent. (?)

426. Symmetry of a Single Figure.

1. The intersection of two planes of symmetry of a solid is an axis of symmetry. (?)

2. The intersections of three planes of symmetry of a solid are three axes of symmetry, and the common intersection of these axes is the center of symmetry. (?)

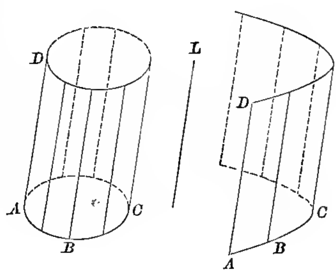
BOOK VII.

I. THE CYLINDER.

427. Definitions and Illustrations.

1. A **cylindrical surface** is a surface generated by a moving straight line which continually touches a given curve and has all of its positions parallel to a given straight line not in the plane of the curve.

Thus, the surface *ABCD*, generated by the moving line *AD* continually touching the curve *ABC*, and always parallel to a given straight line *L*, is a cylindrical surface.



2. The moving line is called the *generatrix;* the curve which directs the motion of the generatrix is called the *directrix;* the generatrix in any position is called an *element* of the surface.

3. The generatrix may be indefinite in extent, and the directrix a closed curve, as in the first diagram, or an open

curve, as in the second. The only cylindrical surfaces considered in this treatise are those whose directrices are closed curves.

4. A **cylinder** is a solid bounded by a cylindrical surface and two parallel planes.

5. The **lateral surface** of a cylinder is its cylindrical surface.

6. The **bases** of a cylinder are the plane portions of its surface.

7. The **axis** of a cylinder is the straight line joining the centers of its bases.

8. The **altitude** of a cylinder is the perpendicular distance from one base to the plane of the other.

9. A **section** of a cylinder is a plane figure whose boundary is the intersection of its plane with the surface of the cylinder.

10. A **right section** of a cylinder is a section perpendicular to the elements.

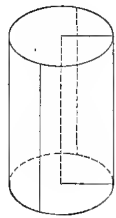
11. A **circular cylinder** is a cylinder whose base is a circle.

12. The **radius** of a circular cylinder is the radius of the base.

13. A **right cylinder** is a cylinder whose elements are perpendicular to its bases. Any element of a right cylinder is equal to its altitude.

14. An **oblique cylinder** is a cylinder whose elements are oblique to its bases. Any element of an oblique cylinder is greater than its altitude.

15. A **cylinder of revolution** is a cylinder generated by the revolution of a rectangle about one side as an axis.



The axis of revolution is called the axis of the cylinder; the side opposite generates the lateral surface; the sides adjacent to the axis generate the bases; the radius of either base is the radius of the cylinder.

16. **Similar cylinders of revolution** are cylinders generated by similar rectangles revolving about homologous sides.

17. **Conjugate cylinders** are cylinders generated by the successive revolutions of a rectangle about two adjacent sides.

18. A **tangent line** to a cylinder is a line having only one point in common with the surface.

19. A **tangent plane** to a cylinder is a plane embracing an element of the cylinder without cutting its surface.

20. The element embraced by a tangent plane is called the *element of contact.*

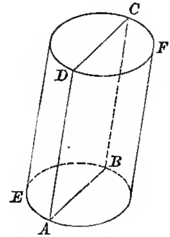
21. A prism is *inscribed* in a cylinder when its lateral edges are elements of the cylinder, and its bases inscribed in the bases of the cylinder.

22. A prism is *circumscribed* about a cylinder when its lateral faces are tangent to the cylinder, and its bases circumscribed about the bases of the cylinder.

428. Proposition I.—Theorem.

Every section of a cylinder embracing an element embraces another element and is a parallelogram.

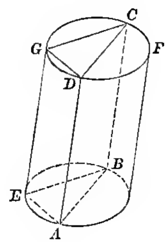
Let $ABCD$ be a section embracing AD, an element of the cylinder EF. Then it embraces another element; for, the element through B is in the cylindrical surface, and, since it is parallel to AD, it is also in the plane of the section, (?) and hence, is the intersection BC of the cylindrical surface and the section AC. Now, AD, BC, are parallel; also, AB, DC; (?) therefore, the section is a parallelogram. If the cylinder is right, the section AC is a rectangle.



429. Proposition II.—Theorem.

The bases of a cylinder are equal.

Any sections, AC, AG, embracing AD, an element of the cylinder EF, are parallelograms; (?) therefore, AB, DC, are equal; also, AE, DG. Now, BC, EG, are equal and parallel; (?) $\therefore$ $BE = CG$; (?) hence, the triangles, ABE, DCG, are equal. (?)



Let one base be applied to the other so that the equal triangles, ABE, DCG, coincide—B coinciding with C, and E with G. Now, B, C, are in the same element; so also are E, G; therefore, any point in the perimeter of either base will coincide with the point in the same element in the other base. Hence, the bases coincide and are therefore equal.

430. Corollaries.

1. *Any two parallel sections cutting all the elements of a cylindrical surface are equal.*

For, these sections are the bases of the cylinder included between them.

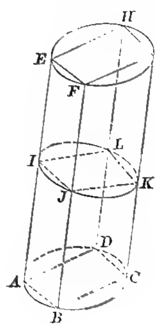
2. *Any section of a cylinder parallel to the base is equal to the base.* (?)

3. *Any plane embracing an element and the center of either base of a cylinder, embraces the center of the other base, the axis, and the centers of all sections parallel to the base.* (?)

4. *The axis of a cylinder passes through the centers of all sections parallel to the bases.* (?)

431. Proposition III.—Theorem.

The lateral area of a cylinder is equal to the product of the perimeter of a right section by an element.



Let l denote the lateral area of the cylinder AG; p, the perimeter of a right section IK; e, an element of the cylindrical surface; l', the lateral area of the inscribed prism BH; p', the perimeter of a right section of the prism; e, an edge which is equal to an element.

Then, $l' = p' \times e$ (**379**).

Now, let the number of faces of the inscribed prism be indefinitely increased, the new edges continually bisecting the arcs in the right section; then, p' increases and approaches p as its limit, l' increases and approaches l as its limit, and e is constant. But, whatever be the number of faces,

$$l' = p' \times e; \quad \therefore \quad l = p \times e \quad (\textbf{301}).$$

432. Corollaries.

1. *The lateral area of a right cylinder is equal to the product of the perimeter of its base by its altitude.* (?)

2. If s denote the entire area, b the area of either base, then,

$$s = p \times e + 2b.$$

3. If l denote the lateral area, s the entire area, r the radius, a the altitude of a cylinder of revolution, then,

$$l = 2\pi ra, \quad (?) \qquad s = 2\pi r\,(a + r). \quad (?)$$

4. *The lateral areas or the entire areas of similar cylinders of revolution are proportional to the squares of their radii or to the squares of their altitudes.*

Let l, l', denote the lateral areas; s, s', the entire areas; r, r', the radii; a, a', the altitudes, of two similar cylinders of revolution; then,

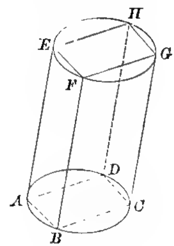
$$\left.\begin{aligned} &\frac{l}{l'} = \frac{2\pi r\,a}{2\pi r'a'} = \frac{r}{r'} \times \frac{a}{a'}. \\ &\text{But,} \quad \frac{r}{r'} = \frac{a}{a'}. \quad (?) \end{aligned}\right\} \quad \therefore \ \frac{l}{l'} = \frac{r^2}{r'^2} = \frac{a^2}{a'^2}. \quad (?)$$

$$\left.\begin{aligned} &\frac{s}{s'} = \frac{2\pi r\,(a + r)}{2\pi r'(a' + r')} = \frac{r}{r'} \times \frac{a + r}{a' + r'}. \\ &\text{But,} \quad \frac{r}{r'} = \frac{a}{a'} = \frac{a + r}{a' + r'}. \quad (?) \end{aligned}\right\} \quad \therefore \ \frac{s}{s'} = \frac{r^2}{r'^2} = \frac{a^2}{a'^2}. \quad (?)$$

433. Proposition IV.—Theorem.

The volume of a cylinder is equal to the product of its base by its altitude.

Let C denote the volume of the cylinder AG; b, its base; a, its altitude; P, the volume of the inscribed prism BH; b', its base; a, its altitude, which is equal to that of the cylinder; then,



$$P = b' \times a \quad (\mathbf{397}).$$

Now, let the number of faces of the inscribed prism be indefinitely increased, the new edges continually bisecting the arcs in the bases; then, b' increases and approaches b as its limit, P increases and approaches C as its limit, and a is constant. But, whatever be the number of faces,

$$P = b' \times a; \quad \therefore \quad C = b \times a \quad (\mathbf{301}).$$

434. Corollaries.

1. If the base of a cylinder is a circle whose radius is r, then,

$$C = \pi r^2 a.$$

2. *The volumes of similar cylinders of revolution are proportional to the cubes of their radii or to the cubes of their altitudes.*

Let C, C', denote the volumes; r, r', the radii; a, a', the altitudes, of two similar cylinders of revolution; then,

$$\left.\begin{aligned} &\frac{C}{C'} = \frac{\pi r^2 a}{\pi r'^2 a'} = \frac{r^2}{r'^2} \times \frac{a}{a'}. \\ &\text{But,} \qquad \frac{r}{r'} = \frac{a}{a'}. \quad (?) \end{aligned}\right\} \quad \therefore \quad \frac{C}{C'} = \frac{r^3}{r'^3} = \frac{a^3}{a'^3}.$$

435. Exercises.

1. How can a cylindrical surface be generated by taking the generatrix for the directrix and the directrix for the generatrix?

2. What is the ratio of the lateral areas, the entire areas, and the volumes of the two conjugate cylinders generated by a rectangle whose base is b and altitude a?

3. The plane embracing an element of a circular cylinder and the tangent line to the base at the intersection of its circumference with the element, is tangent to the cylinder.

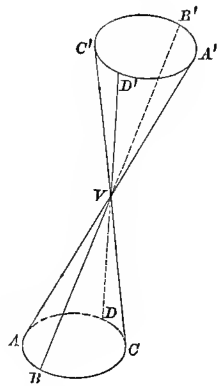
4. The intersection of two tangent planes to a cylinder is parallel to the elements.

II. THE CONE.

436. Definitions and Illustrations.

1. A **conical surface** is a surface generated by a moving straight line continually touching a given curve and passing through a fixed point not in the plane of the curve.

Thus, the surface generated by the moving line AA', continually touching the curve $ABCD$, and passing through the fixed point V, is a conical surface.



2. The moving line is called the *generatrix*; the curve which directs its motion, the *directrix*; and any position of the generatrix, an *element*.

3. A conical surface generated by an indefinite line consists of two portions called *nappes*, — one the *lower nappe*, the other the *upper nappe*.

4. A **cone** is a solid bounded by a conical surface and a plane.

5. The **lateral surface** of a cone is its conical surface.

6. The **base** of a cone is the plane portion of its surface.

7. The **vertex** of a cone is the fixed point through which all the elements pass.

8. The **altitude** of a cone is the perpendicular distance from its vertex to the plane of its base.

9. The **axis** of a cone whose base has a center is the line joining that center with the vertex.

10. A **section** of a cone is a plane figure whose boundary is the intersection of its plane with the surface of the cone.

11. A **right section** of a cone is a section perpendicular to the axis.

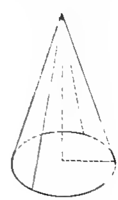
12. A **circular cone** is a cone whose base is a circle.

13. A **right cone** is a cone whose axis is perpendicular to its base. The axis of a right cone is equal to its altitude.

14. An **oblique cone** is a cone whose axis is oblique to its base. The axis of an oblique cone is greater than its altitude.

15. A **cone of revolution** is a cone generated by the revolution of a right triangle about one of its perpendicular sides as an axis.

16. The side about which the triangle revolves is called the *axis* of the cone; the other perpendicular side generates the *base*; the hypotenuse generates the *conical surface*; any position of the hypotenuse is an *element*; any element is called the *slant height*.



17. **Similar cones of revolution** are cones generated by

the revolution of similar right triangles about homologous perpendicular sides.

18. **Conjugate cones** are cones generated by the successive revolutions of a right triangle about its perpendicular sides.

19. A **truncated cone** is the portion of a cone included between the base and a section cutting all the elements.

20. A **frustum** of a cone is a truncated cone in which the cutting section is parallel to the base.

21. The base of the cone is called the *lower base* of the frustum, and the parallel section the *upper base.*

22. The **altitude** of a frustum is the perpendicular distance from one base to the plane of the other.

23. The **lateral surface** of a frustum is the portion of the lateral surface of the cone included between the bases of the frustum.

24. The **slant height** of a frustum of a cone of revolution is the portion of any element of the cone included between the bases.

25. A **tangent line** to a cone is a line having only one point in common with the surface.

26. A **tangent plane** to a cone is a plane embracing an element of the cone without cutting the surface.

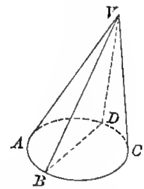
The element embraced by a tangent plane is called the *element of contact.*

27. A pyramid is *inscribed* in a cone when its lateral edges are elements of the cone and its base is inscribed in the base of the cone.

28. A pyramid is *circumscribed* about a cone when its lateral faces are tangent to the cone and its base is circumscribed about the base of the cone.

437. Proposition V.—Theorem.

Every section of a cone through its vertex is a triangle.

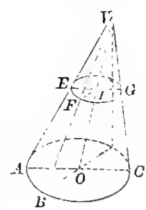


Let VBD be a section of the cone $V\text{-}ABC$ through the vertex V; then, VBD is a triangle.

For, the straight lines joining V, B, and V, D, are elements of the surface; (?) they also lie in the plane of the section; (?) hence, they are the intersections of the conical surface with the plane of the section. BD is also a straight line; (?) therefore the section VBD is a triangle.

438. Proposition VI.—Theorem.

Every section of a circular cone parallel to the base is a circle.



The section EFG of the circular cone $V\text{-}ABC$, parallel to the base, is a circle.

For, let O be the center of the base, and I the point in which the axis VO pierces the parallel section.

Planes embracing the axis VO and any elements, as VA, VB, intersect the base in the radii, OA, OB, and the parallel section in IE, IF, which are respectively parallel to OA, OB. (?)

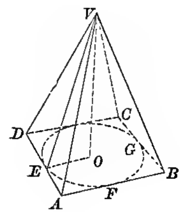
$$\therefore \left.\begin{cases} IE : OA :: VI : VO. \\ IF : OB :: VI : VO. \end{cases}\right\} (?) \quad \therefore \quad IE : OA :: IF : OB.$$

But, $\qquad OA = OB; \quad \therefore \quad IE = IF.$

Hence, any two straight lines drawn from I to the perimeter of the section are equal; therefore the section is a circle whose center is I.

439. Proposition VII.—Theorem.

The lateral area of a cone of revolution is equal to one-half the product of the circumference of its base by its slant height.



Let l denote the lateral area of the cone of revolution $V\text{-}EFG$; c, the circumference of its base; h, the slant height, which is equal to an element of the cone; l', the lateral area of the regular circumscribed pyramid $V\text{-}ABCD$; p, the perimeter of its base; h, its slant height, which is equal to the slant height of the cone. Then, $l' = \frac{1}{2}p \times h$.

Now, let the number of faces of the circumscribed pyramid be indefinitely increased, the new elements of contact continually bisecting the arcs in the base; then, p diminishes and approaches c as its limit, l' diminishes and approaches l as its limit, and h is constant. But, whatever be the number of faces of the pyramid,

$$l' = \tfrac{1}{2}p \times h; \quad \therefore \quad l = \tfrac{1}{2}c \times h. \quad (?)$$

440. Corollaries.

1. If l denote the lateral area, s the entire area, r the radius, h the slant height of a cone of revolution, then,

$$l = \pi r h, \qquad s = \pi r(h + r). \quad (?)$$

2. *The lateral areas or the entire areas of similar cones of revolution are proportional to the squares of their radii or to the squares of their slant heights.* (?) (**432**, 4).

3. *The lateral area of a frustum of a cone of revolution is equal to one-half the product of the sum of the circumferences of its bases by its slant height.*

Let l denote the lateral area of the frustum; c, c', respectively, the circumferences of the lower and upper bases; r, r', the radii of the bases; h', h, $h'-h$, the slant heights of the cone, the frustum, and the part above the frustum.

Then, $l = \pi rh' - \pi r'(h'-h) = \pi(rh' - r'h' + r'h)$.

But, $r : r' :: h' : h'-h$; (?) $\therefore$ $rh' - rh = r'h'$.

$\therefore$ $rh' - r'h' = rh$; $\therefore$ $l = \pi(rh + r'h)$.

$\therefore$ $l = \pi(r + r')h = \frac{1}{2}(c + c')h$. (?)

4. *The lateral area of a frustum of a cone of revolution is equal to the product of the circumference of a section equidistant from its bases by its slant height.*

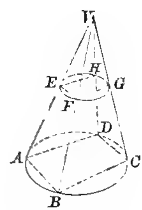
Let r'' denote the radius of this section; c'', its circumference; r, r', the radii of the bases. Then,

$$r'' = \tfrac{1}{2}(r + r'); \text{ (?) } \therefore \quad 2r'' = r + r'.$$

$$\therefore \quad l = 2\pi r''h = c''h.$$

441. Proposition VIII.—Theorem.

Any section of a cone parallel to the base is to the base as the square of the altitude of the part above the section is to the square of the altitude of the cone.



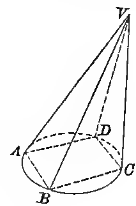
Let b denote the base of the cone V-AC; a, its altitude; b', the section EG of the cone, parallel to the base; a', the altitude of the cone above the section; b_1, the base of the inscribed pyramid; b_2, the section $EFGH$ of the pyramid; a, a', the altitudes of the pyramid and the part above the section. Then, $b_2 : b_1 :: a'^2 : a^2$ (**402,** 1).

Now, let the number of faces of the inscribed pyramid be indefinitely increased, the new edges continually bisecting the arcs in the base of the cone; then, b_1, b_2, increase and approach their respective limits, b, b'. But, whatever be the number of faces of the pyramid,

$$b_2 : b_1 :: a'^2 : a^2; \quad \therefore \quad b' : b :: a'^2 : a^2. \quad (?)$$

442. Proposition IX.—Theorem.

The volume of a cone is equal to one-third of the product of its base by its altitude.



Let C denote the volume of the cone V-AC; b, its base; a, its altitude; P, the volume of the inscribed pyramid V-$ABCD$; b', its base; a, its altitude, which is equal to the altitude of the cone. Then,

$$P = \tfrac{1}{3}b' \times a.$$

Now, let the number of faces of the inscribed pyramid be indefinitely increased, the new edges continually bisecting the arcs in the base of the cone; then, b', P, increase and approach their respective limits, b, C. But, whatever be the number of faces of the pyramid,

$$P = \tfrac{1}{3}b' \times a; \quad \therefore \quad C = \tfrac{1}{3}b \times a.$$

443. Corollaries.

1. *Any two cones are proportional to the products of their bases by their altitudes;* (?) *if their bases are equivalent, they are proportional to their altitudes;* (?) *if their altitudes are equal, they are proportional to their bases.* (?)

2. If C, r, a, respectively denote the volume, radius, and altitude of a circular cone,

$$C = \tfrac{1}{3}\pi r^2 a. \quad (?)$$

3. *Similar cones of revolution are proportional to the cubes of their altitudes or to the cubes of their radii.* (?)

4. *The volume of a frustum of any cone is equal to one-third of the product of its altitude by the sum of its lower base, its upper base, and a mean proportional between its bases.*

Let F denote the volume of the frustum; b, b', its bases; a', a, $a' - a$, the altitudes of the cone, the frustum, and the part above the frustum. Then, as in **407**, 4, let the student develop the formula,

$$F = \tfrac{1}{3} a (b + b' + \sqrt{bb'}).$$

5. If F denote the volume of a frustum of a cone of revolution, a its altitude, r, r', the radii of its bases,

$$b = \pi r^2, \quad b' = \pi r'^2, \quad \sqrt{bb'} = \pi r r'.$$

$$\therefore \quad F = \tfrac{1}{3} \pi a (r^2 + r'^2 + r r').$$

444. Exercises.

1. What is the ratio of the lateral areas, the entire areas, and the volumes of the two conjugate cones generated by a triangle whose base is b and altitude a?

2. The plane embracing an element of a circular cone and the tangent line to the base at the intersection of its circumference with the element is tangent to the cone.

3. A section of any cone parallel to the base is similar to the base.

4. How many barrels of water will that cistern contain the altitude of which is 8 ft., the diameter at the bottom 4 ft., and at the top 6 ft.? *Ans.* 37.8 bbl.

5. The volume of a frustum of a cone of revolution is 7050 cu. in.; its altitude, 12 in.; the diameter of the lower

base twice that of the upper base. What are the diameters of the bases? *Ans.* 35.8096 in., 17.9048 in.

6. If a frustum of a cone of revolution of which the altitude is 10 ft., and the diameters of the bases 6 ft., 4 ft., respectively, be divided into two equal parts by a plane parallel to its bases, what would be the altitude of each part? *Ans.* 4.0375 ft., 5.9625 ft.

III. THE SPHERE.

I. Sections and Tangents.

445. Definitions.

1. A **sphere** is a solid bounded by a surface all of whose points are equally distant from a point within, called the *center*. A sphere may be generated by the revolution of a semicircle about its diameter as an axis.

2. A **radius** of a sphere is the distance from the center to any point of the surface. All the radii of a sphere are equal. (?)

3. A **diameter** of a sphere is any straight line through the center whose extremities are in the surface. All the diameters of a sphere are equal, since each is twice the radius.

4. A **plane section** of a sphere is a plane figure whose boundary is the intersection of its plane with the surface of the sphere.

5. A line or plane is *tangent to a sphere* when it has only one point in common with the surface of the sphere.

6. Two spheres are *tangent to each other* when their surfaces have only one point in common.

7. A polyhedron is *circumscribed about a sphere* when all of its faces are tangent to the sphere. In this case, the sphere is inscribed in the polyhedron.

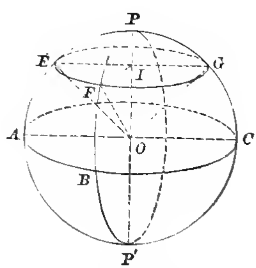
8. A polyhedron is *inscribed in a sphere* when all its vertices are in the surface of the sphere. In this case, the sphere is circumscribed about the polyhedron.

9. A cylindrical or conical surface is *circumscribed about a sphere* when all its elements are tangent to the sphere.

10. A cylinder or a cone is *circumscribed about a sphere* when its bases and cylindrical surface, or the base and conical surface, are tangent to the sphere. In this case, the sphere is inscribed in the cylinder or cone.

446. Proposition X.—Theorem.

Every plane section of a sphere is a circle.



The plane section EFG of a sphere whose center is O is a circle.

For, from the center O draw OI perpendicular to the section, and the radii, OE, OF, OG, . . . to different points of the boundary of the section.

The radii, OE, OF, OG, . . . are equal; (?) therefore, IE, IF, IG, . . . are equal; (?) hence, the section EFG is a circle whose center is I, the foot of the perpendicular OI.

447. Corollaries and Definitions.

1. *The line joining the centers of a sphere and a circle of the sphere is perpendicular to the circle.* (?)

2. If r, r', p, respectively denote the radius of a sphere, the radius of a circle of the sphere, and the perpendicular from the center of the sphere to the circle, then,

$$r' = \sqrt{r^2 - p^2}.$$

3. *All circles of a sphere equally distant from the center are equal.* (?)

4. As p increases to r, r' diminishes to 0; and as p diminishes to 0, r' increases to r; hence,

5. A **great circle** of a sphere is a circle passing through the center.

6. A **small circle** of a sphere is a circle not passing through the center.

7. *All great circles of a sphere are equal.* (?)

8. *Any two great circles of a sphere bisect each other.* (?)

9. *Any great circle of a sphere bisects the sphere.* (?)

10. *An arc of a great circle can be drawn through any two points on the surface of a sphere*; (?) *and only one, if the points are not at the extremities of a diameter*; (?) *but an indefinite number, if the points are at the extremities of a diameter.* (?)

11. *An arc of a circle can be drawn through any three points on the surface of a sphere.* (?)

12. An **axis** of a circle of a sphere is the diameter of the sphere perpendicular to the circle.

13. The **poles** of a circle of a sphere are the extremities of its axis.

14. *A pole of a circle of a sphere is equally distant from all the points in the circumference of the circle.* (?)

15. *The poles of a great circle are equally distant from its circumference.* (?)

16. *The poles of a small circle are unequally distant from its circumference.* (?)

17. *All arcs of great circles drawn from a pole of a circle to points in the circumference of the circle are equal.* (?)

18. The **distance** on the surface of a sphere, from one point of the surface to another, is usually understood to be the arc of a great circle having the points for its extremities.

19. The **polar distance** of a circle of a sphere is the arc of a great circle drawn from any point of the circumference of the circle to its nearest pole.

20. *The polar distance of a great circle is a quadrant of a great circle.* (?)

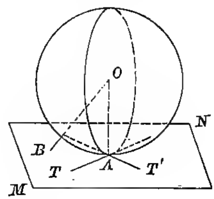
21. *The point on the surface of a sphere which is at a quadrant's distance from two points, not at the extremities of a diameter, in an arc of a great circle, is the pole of the arc.* (?)

22. By means of these properties, arcs of great or small circles can be drawn with great facility on a spherical blackboard, thus:

23. To draw an arc of a great circle through two given points on the surface of a sphere, describe about each of the points, as a pole, with an arc of a great circle equal to a quadrant, the arc of a great circle; then, the great circle described about the point of intersection of these arcs, as a pole, with an arc of a great circle equal to a quadrant, will pass through the two points.

448. Proposition XI.—Theorem.

A plane perpendicular to a radius at its extremity is tangent to the sphere at that point.



Let O be the center of a sphere, and MN a plane perpendicular to the radius OA at its extremity A; then, MN is tangent to the sphere at A.

For, to any other point B, in the plane MN, draw the straight line OB. Since OA is perpendicular to MN, OB is oblique; (?) $\therefore$ $OB > OA$; (?) hence, B is without the sphere; (?) therefore, the plane MN has only one point in common with the sphere; (?) hence, MN is tangent to the sphere at A.

449. Corollaries.

1. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.* (?)

2. *A straight line tangent to a circle of a sphere lies in the plane tangent to the sphere at the point of contact.* (?)

3. *Any straight line in a tangent plane through the point of contact is tangent to the sphere at that point.* (?)

4. *The plane of any two straight lines tangent to a sphere at the same point is tangent to the sphere at that point.* (?)

450. Exercises.

1. The intersection of the surfaces of two spheres is the circumference of a circle, and the straight line joining the centers of the spheres is perpendicular to the circle at its center.

2. Given a material sphere, to find its radius.

3. Pass a plane tangent to a given sphere and embracing a given straight line without the sphere.

4. Through any four points not in the same plane one spherical surface can pass, and only one.

5. The four perpendiculars to the planes of the faces of a tetrahedron, at their centers, meet at the same point.

6. The six planes perpendicular to the edges of a tetrahedron, at their middle points, meet in the same point.

7. A sphere can be inscribed in any tetrahedron.

8. The six planes bisecting the dihedral angles of a tetrahedron meet in the same point.

9. The shortest distance on the surface of a sphere from one point of the surface to another is the arc of a great circle, not greater than a semi-circumference, whose extremities are these points.

II. Spherical Angles and Triangles.

451. Definitions and Remarks.

1. The **angle of two curves** having a common point is the angle of their tangents at that point.

2. A **spherical angle** is the angle of two arcs of great circles of a sphere.

3. A **spherical polygon** is a portion of the surface of a sphere bounded by three or more arcs of great circles.

4. The **sides** of a spherical polygon are the bounding arcs; the **angles** are the angles included by consecutive sides; the **vertices** are the intersections of the sides.

5. A **diagonal** of a spherical polygon is an arc of a great circle dividing the polygon and terminating in two vertices not consecutive.

6. The planes of the sides of a spherical polygon form by their intersections a polyhedral angle whose vertex is the center of the sphere, and whose facial angles are measured by the sides of the polygon.

7. A **spherical pyramid** is a portion of a sphere bounded by a spherical polygon and the planes of the sides of the polygon. The spherical polygon is the *base* of the pyramid, and the center of the sphere is its *vertex*.

A spherical pyramid is designated according to its base, as *triangular*, *quadrangular*, . . .

8. A **convex spherical polygon** is a spherical polygon whose corresponding polyhedral angle is convex.

The only spherical polygons here considered are those in which no side exceeds a semi-circumference.

9. A **spherical triangle** is a spherical polygon of three sides.

10. A spherical triangle, like a plane triangle, is *right* or *oblique*, *scalene*, *isosceles*, or *equilateral*.

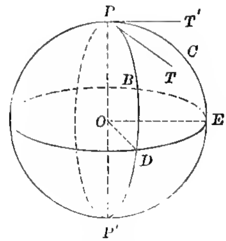
11. Two spherical triangles are *symmetrical*, if their successive sides and angles taken in a reverse order are equal.

12. Two spherical pyramids are *symmetrical*, if their bases are symmetrical triangles. (?)

13. The **polar** of a spherical triangle is a spherical triangle the poles of whose sides are respectively the vertices of the given triangle.

452. Proposition XII.—Theorem.

A spherical angle measures the dihedral angle included by the planes of its sides.



By definition, the spherical angle BPC is the angle TPT' of the tangents to its sides, BP, CP, at their point of intersection P.

Since the tangent to an arc lies in the plane of that arc, and is perpendicular to the diameter drawn to the point of contact, the sides, TP, $T'P$, of the angle TPT', respectively, lie in the planes of the sides of the spherical angle BPC, and are both perpendicular, at the same point, to the diameter PP', which is the edge of the dihedral angle included by the planes of these sides, since they are arcs of great circles. Hence, the angle TPT' measures the dihedral angle included by the planes of the sides, BP, CP.

453. Corollaries.

1. *The angles of a spherical polygon respectively measure the dihedral angles of the corresponding spherical pyramid.* (?)

2. *A spherical angle is measured by the arc of a great circle described about its vertex as a pole, and intercepted by its sides produced, if necessary.*

For, if DE is the arc of a great circle described about the vertex P as a pole, the arcs, PD, PE, are quadrants; hence, DO, EO, are perpendicular to PP'; (?) therefore, DOE measures the dihedral angle $BPOC$, and hence, the spherical angle BPC. Therefore, the arc DE, which measures the angle DOE, measures the spherical angle BPC.

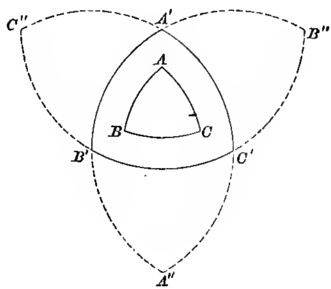
3. *The angle included by two arcs of small circles is the same as the spherical angle having the same vertex, and whose sides have the same tangents at the vertex.* (?)

4. *Two circumferences of great circles of a sphere intersect at right angles if either passes through the pole of the other, and conversely.* (?)

5. *The arc of a great circle described about a point on a given arc of a great circle as a pole, passes through all the points at a quadrant's distance from the pole, and is perpendicular to the given arc.* (?)

454. Proposition XIII.—Theorem.

If one triangle is the polar of another, the second is the polar of the first.



The triangle $A'B'C'$, whose sides are arcs of great circles described about the vertices, A, B, C, as poles, is, by definition, the polar of the triangle ABC; then, ABC is the polar of $A'B'C'$.

For, since B is the pole of the arc $A'C'$, and C the pole of the arc $A'B'$, A' is at a quadrant's distance from each of the points B, C, and is, therefore, the pole of the arc BC.

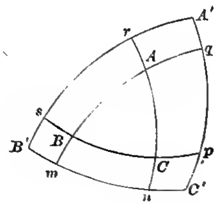
Also, B' is the pole of the arc AC, and C' the pole of the arc AB; hence, ABC is the polar of $A'B'C'$.

Scholium.—The arcs of great circles described about A, B, C, as poles, will, if sufficiently produced, form three triangles exterior to the polar.

The polar triangles are distinguished by having their homologous vertices, A, A', on the same side of BC and of $B'C'$; B, B', on the same side of AC and of $A'C'$; C, C', on the same side of AB and of $A'B'$.

455. Proposition XIV.—Theorem.

Each angle in either of two polar triangles is the supplement of the side opposite in the other, and each side is the supplement of the angle opposite in the other.



Let ABC, $A'B'C'$, be polar triangles in which a, b, c, and a', b', c', respectively, denote the sides opposite the angles A, B, C, and A', B', C'.

Since B' is the pole of the arc AC, and C' the pole of the arc AB,

$$B'n = 90^\circ, \quad mC' = 90^\circ;$$

$$\therefore \quad B'n + mC' = 180^\circ.$$

But, $\quad B'C' + mn = B'm + mn + nC' + mn,$

or, $\quad a' + mn = B'n + mC' = 180^\circ;$

$$\therefore \quad mn = 180^\circ - a'.$$

But mn measures the angle A; (?)

$$A = 180^\circ - a'; \quad \therefore \quad a' = 180^\circ - A.$$

Also, $\quad B = 180^\circ - b'; \quad \therefore \quad b' = 180^\circ - B.$

$$C = 180^\circ - c'; \quad \therefore \quad c' = 180^\circ - C.$$

$$A' = 180^\circ - a; \quad \therefore \quad a = 180^\circ - A'.$$

$$B' = 180^\circ - b; \quad \therefore \quad b = 180^\circ - B'.$$

$$C' = 180^\circ - c; \quad \therefore \quad c = 180^\circ - C'.$$

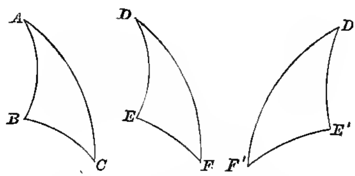
In consequence of these relations, polar triangles are also called supplemental triangles.

456. Corollary.

The trihedral angles formed by the radii drawn from the center of the sphere to the vertices of the angles of two polar triangles are supplemental. (?) (**358,** 24), (**451,** 6), (**453,** 1).

457. Proposition XV.—Theorem.

Two triangles on the same sphere or on equal spheres, having two sides and the included angle of one respectively equal to two sides and the included angle of the other, are either equal or symmetrical.



1. Let the two sides, AB, AC, and the included angle A, of the triangle ABC, be respectively equal to the two sides, DE, DF, and the included angle D, of the triangle DEF, and arranged in the same order. Then, these triangles are equal, for they can be applied so as to coincide. (?) (**55**).

2. If the equal parts are arranged in a reverse order, as in the triangles ABC, $D'E'F'$, let DEF be the symmetrical triangle of $D'E'F'$; then, ABC and DEF are equal, since they can be made to coincide; hence, ABC and $D'E'F'$ are symmetrical.

458. Proposition XVI.—Theorem.

Two triangles on the same sphere or on equal spheres, having a side and two adjacent angles of one respectively equal to a side and two adjacent angles of the other, are either equal or symmetrical. (?) (**57**).

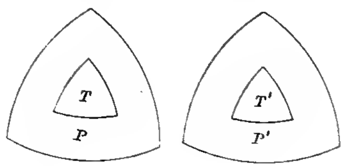
459. Proposition XVII.—Theorem.

Two mutually equilateral triangles on the same sphere or on equal spheres are mutually equiangular, and either equal or symmetrical.

For, the facial angles of the corresponding trihedral angles at the center of the sphere are respectively equal, since they are measured by equal sides of the triangles; hence, the dihedral angles are respectively equal (**373**); consequently, the angles of the triangles are respectively equal; therefore, the triangles are equal or symmetrical, according as their equal sides are arranged in the same or in a reverse order.

460. Proposition XVIII.—Theorem.

Two mutually equiangular triangles on the same sphere or on equal spheres are mutually equilateral, and either equal or symmetrical.



Let T, T', be mutually equiangular spherical triangles; P, P', their respective polars.

Since T, T', are mutually equiangular, P, P', are mutually equilateral; (?) therefore, P, P', are mutually equiangular; (?) hence, T, T', are mutually equilateral; (?) consequently, T, T', are either equal or symmetrical. (?)

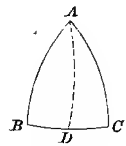
461. Corollaries.

1. *Two mutually equiangular triangles on unequal spheres are not mutually equilateral, but their sides are respectively proportional to the radii of the spheres.* (?)

2. *Either of two mutually equiangular triangles on unequal spheres is similar to the other or to the symmetrical of the other, according as the equal angles are arranged in the same or in a reverse order.* (?)

462. Proposition XIX.—Theorem.

The angles opposite the equal sides of an isosceles spherical triangle are equal.



In the isosceles spherical triangle ABC, let $AB = AC$; then, $B = C$.

For, the arc AD of a great circle drawn from the vertex A to the middle point D of the base BC, divides the triangle ABC into two triangles, ABD, ACD, which are mutually equilateral, and hence, mutually equiangular; (?) hence, B and C are equal, since they are opposite the common side AD.

463. Corollaries.

1. *The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle point of the base bisects the vertical angle, is perpendicular to the base, and divides the triangle into two symmetrical triangles.* (?)

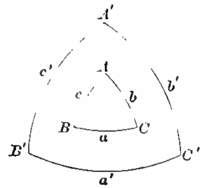
2. *Symmetrical isosceles spherical triangles are equal.* (?)

464. Proposition XX.—Theorem.

If two angles of a spherical triangle are equal, the sides opposite these angles are equal, or the triangle is isosceles.

In the spherical triangle ABC, let $B = C$; then, $AC = AB$.

For, let $A'B'C'$ be the polar triangle of ABC, and denote the sides of the triangles respectively opposite the angles, A, B, C, and A', B', C', by a, b, c, and a', b', c'; then,



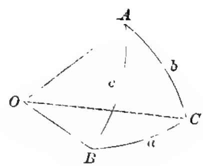
$$\left.\begin{aligned} b' &= 180° - B; \\ c' &= 180° - C; \end{aligned}\right\} \text{ but, } B = C;$$

$$\therefore\ b' = c';\ \therefore\ B' = C'.$$

$$\left.\begin{aligned} b &= 180° - B'; \\ c &= 180° - C'; \end{aligned}\right\} \text{ but, } B' = C';\ \therefore\ b = c, \text{ or } AC = AB.$$

465. Proposition XXI.—Theorem.

Any side of a spherical triangle is less than the sum of the other sides, and greater than their difference.



Let ABC be a spherical triangle on the sphere whose center is O.

Since any facial angle of the corresponding trihedral angle is less than the sum of the other two, the side of the triangle which measures it is less than the sum of the sides which measure the other facial angles.

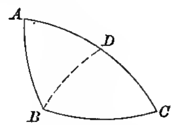
$$\therefore\ \begin{cases} a < b + c; & \therefore\ b > a - c, \quad c > a - b. \\ b < a + c; & \therefore\ a > b - c, \quad c > b - a. \\ c < a + b; & \therefore\ a > c - b, \quad b > c - a. \end{cases}$$

466. Corollary.

Any side of a spherical polygon is less than the sum of the other sides. (?)

467. Proposition XXII.—Theorem.

In a spherical triangle, the greater side is opposite the greater angle, and the greater angle is opposite the greater side.



1. In the triangle ABC, let $B > C$; then, $AC > AB$. For, draw the arc BD of a great circle, making the angle $CBD = BCD$; then, $DC = DB$. (?)

$AD + DB > AB$; $\therefore$ $AD + DC > AB$; $\therefore$ $AC > AB$.

2. Let $AC > AB$; then, $B > C$. For, if $B = C$, $AC = AB$; if $B < C$, $AC < AB$. But these results are contrary to the hypothesis; hence, B is neither equal to C nor less than C; $\therefore$ $B > C$.

468. Proposition XXIII.—Theorem.

The sum of the sides of a spherical triangle can have any value between the limits 0° *and* 360°.

For, these sides respectively measure the facial angles of the corresponding trihedral angle at the center of the sphere; but the sum of these facial angles can have any value between the limits 0° and 360° (**371**); hence, the sum of the sides can have any value between these limits.

469. Corollaries and Definitions.

1. *The sum of the angles of a spherical triangle can have any value between the limits two right angles and six right angles.*

For, employing the usual notation for a spherical triangle and its polar,

$$A = 180° - a', \quad B = 180° - b', \quad C = 180° - c'.$$

$$\therefore \quad A + B + C = 540° - (a' + b' + c').$$

But $a' + b' + c'$ can have any value between 360° and 0°;
$\therefore$ $A + B + C$ can have any value between 180° and 540°.

2. *One side of a spherical triangle can be a quadrant, two can be quadrants, or three can be quadrants.*

For, one of the facial angles of the corresponding trihedral angle can be a right angle, two can be right angles, or three can be right angles.

3. A spherical triangle is *quadrantal*, *bi-quadrantal*, or *tri-quadrantal*, according as one of its sides is a quadrant, two are quadrants, or three are quadrants.

4. *A spherical triangle can have one right angle, two right angles, or three right angles.*

For, one side of its polar triangle can be a quadrant, two can be quadrants, or three can be quadrants.

5. A spherical triangle is *rectangular*, *bi-rectangular*, or *tri-rectangular*, according as it has one right angle, two right angles, or three right angles.

6. *A bi-quadrantal triangle is bi-rectangular.*

For, the vertex between the quadrants is the pole of the opposite side; hence, the planes of the quadrants embrace the axis of the great circle of which the third side is an arc, and the planes of the quadrants are perpendicular to the plane of the third side; hence, the triangle is bi-rectangular.

7. *A bi-rectangular triangle is bi-quadrantal.*

For, its polar triangle is bi-quadrantal, (?) and hence, bi-rectangular; therefore, the given triangle is bi-quadrantal.

8. *A tri-quadrantal triangle is tri-rectangular.*

For, each angle is measured by its opposite side, which is a quadrant.

9. *A tri-rectangular triangle is tri-quadrantal.*

For, its polar triangle is tri-quadrantal, (?) and hence, tri-rectangular; hence, the given triangle is tri-quadrantal.

10. *A tri-rectangular triangle and its polar are coincident.*

For, each vertex is the pole of the opposite side.

11. *A tri-rectangular triangle is one-eighth of the surface of the sphere.*

For, the three great circles, each of which is perpendicular to the plane of the other two, divide the surface of the sphere into eight tri-rectangular triangles, which are equal, since they can be made to coincide.

12. *One of the sides of a spherical triangle can be greater than a quadrant, two can be greater than quadrants, or three can be greater than quadrants.* (?)

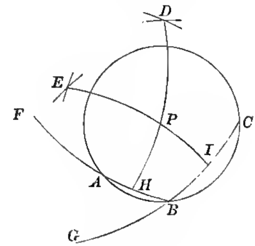
13. *A spherical triangle can have one obtuse angle, two obtuse angles, or three obtuse angles.* (?)

470. Proposition XXIV.—Problem.

To describe the arc of a circle through three given points on the surface of a sphere.

Let A, B, C, be the three given points. About A, B,

as poles, with quadrants, describe arcs intersecting in D; about B, C, as poles, with quadrants, describe arcs intersecting in E.



The arc of a great circle described about D as a pole will pass through A, B. (?) The arc of a great circle described about E as a pole will pass through B, C. The arc DH of a great circle described about the pole F on BA produced, at a quadrant's distance from H, the middle point of AB, is perpendicular to AB (**453,** 5). The arc EI of a great circle described about the pole G on CB produced, at a quadrant's distance from I, the middle point of BC, is perpendicular to BC.

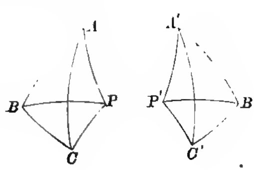
Any point of DH is equally distant from A, B; (?) any point of EI is equally distant from B, C; therefore, P, the intersection of DH, EI, is equally distant from A, B, C; hence, the circle described about P as a pole, with the arc of a great circle equal to the distance from P to one of the points, A, B, C, will pass through A, B, C.

The circle will, in general, be a small circle.

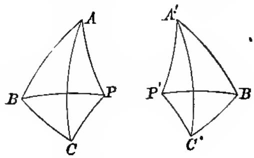
471. Proposition XXV.—Theorem.

Symmetrical spherical triangles are equivalent.

Let ABC, $A'B'C'$, be two symmetrical spherical triangles having $AB = A'B'$, $AC = A'C'$, $BC = B'C'$, $A = A'$, $B = B'$, $C = C'$; then, ABC, $A'B'C'$, are equivalent.



For, let P, P', be the poles of the small circles whose circumferences pass through A, B, C, and A', B', C', respectively.



These small circles are equal, since they are circumscribed about plane triangles whose sides are respectively equal, being chords of equal arcs.

Draw arcs of great circles from the poles, P, P', to the vertices, A, B, C, and A', B', C', respectively.

The triangles, PAB, $P'A'B'$, are symmetrical and isosceles; so also are PBC, $P'B'C'$, and PAC, $P'A'C'$.

$$\therefore \quad PAB = P'A'B', \quad PBC = P'B'C', \quad PAC = P'A'C';$$

$$\therefore \quad PAB + PBC - PAC = P'A'B' + P'B'C' - P'A'C';$$

$$\therefore \quad ABC = A'B'C'.$$

If the poles are within the triangles, the three equations must be added.

472. Corollary.

Two symmetrical spherical pyramids are equivalent.

473. Exercises.

1. Construct a right spherical triangle on a spherical black-board.

2. Construct a tri-rectangular triangle.

3. Construct a spherical triangle having two sides respectively 90°, 45°, and their included angle 60°, and construct its polar.

4. Construct a spherical triangle whose sides are 30°, 60°, 90°, respectively, and describe a circumference through its vertices.

5. The sum of two arcs of great circles drawn from any point within a triangle to the extremities of any side is less than the sum of the other sides.

6. Two spherical triangles whose vertices are respectively the opposite extremities of three diameters are symmetrical.

7. The angle of two intersecting curves on the surface of a sphere is equal to the dihedral angle of the planes embracing the tangents to the two curves at their point of intersection and the center of the sphere.

8. If two spherical triangles on the same sphere or on equal spheres have two sides of the one respectively equal to two sides of the other, and the included angles unequal, the third sides are unequal and the greater third side belongs to the triangle having the greater included angle, and conversely.

III. Measurements of the Sphere.

474. Definitions.

1. A **lune** is a portion of the surface of a sphere bounded by two semi-circumferences of great circles.

2. The **angle** of a lune is the angle included by the semi-circumferences which form its boundary.

3. A **spherical ungula**, or wedge, is a portion of a sphere bounded by a lune and two great semicircles.

4. The **base** of an ungula is the bounding lune.

5. The **angle** of an ungula is the dihedral angle of its bounding semicircles, and is equal to the angle of its bounding lune. (?)

6. The **edge** of an ungula is the edge of its angle.

7. The **spherical excess** of a spherical triangle is the sum of its angles minus two right angles.

8. The spherical excess of a spherical polygon is the sum of its angles, minus two right angles taken as many times less two as the polygon has sides.

9. A **zone** is a portion of the surface of a sphere intercepted by the circumferences of two parallel circles of the sphere.

10. The **bases** of a zone are the circumferences of the intercepting circles.

If the plane of one base becomes tangent to the sphere, that base becomes a point, and the zone is said to have only one base. If the planes of both bases become tangent to the sphere, the bases become points at the extremities of a diameter, and the zone becomes the surface of the sphere.

11. The **altitude** of a zone is the perpendicular distance from one base to the plane of the other.

12. A **spherical segment** is a portion of a sphere bounded by a zone and the two parallel circles whose circumferences are the bases of the zone.

13. The **bases** of a spherical segment are the bounding circles.

A spherical segment has only one base when its bounding zone has only one base, and becomes the sphere when its bounding zone becomes the surface of the sphere.

14. The **altitude** of a spherical segment is the perpendicular distance from one base to the plane of the other.

15. A **spherical sector** is a portion of a sphere generated by a circular sector of the semicircle which generates the sphere.

16. The **bounding surfaces** of a spherical sector are, in

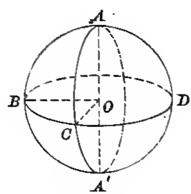
general, three curved surfaces, — two conical surfaces generated by the radii of the generating circular sector, and the zone generated by the arc of the circular sector.

17. The **base** of a spherical sector is the bounding zone.

18. When will one of the conical surfaces of a spherical sector become a line? when a plane? when will one be convex and the other concave? when will both be concave? when will both be lines, and what will the sector be?

475. Proposition XXVI.—Theorem.

A lune is to the surface of the sphere as the angle of the lune is to four right angles.



Let L denote the lune $ABA'CA$, whose angle is A; S, the surface of the sphere; BCD, a great circle whose pole is A; and R, a right angle.

The arc BC measures the angle A of the lune; the circumference BCD measures four right angles.

1. If the circumference BCD and the arc BC are commensurable, suppose the circumference divided into m equal parts, and that n of these parts are contained in the arc BC.

Planes through AA' and the respective points of division divide the surface of the sphere into m equal lunes; (?) the lune L contains n of these lunes.

$$\therefore \left\{ \begin{array}{l} L : S :: n : m; \\ A : 4R :: n : m; \end{array} \right\} \quad \therefore \quad L : S :: A : 4R.$$

2. If the circumference BCD and the arc BC are incommensurable, the proposition can be proved by the method employed in **150**, 2.

476. Corollaries.

1. *Two lunes on the same sphere or on equal spheres are proportional to their angles.* (?)

2. *A lune is equivalent to the product of a tri-rectangular triangle by twice the angle of the lune, the right angle being taken as the unit of angles.*

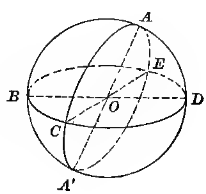
For, let L denote a lune; A, its angle; T, a tri-rectangular triangle; S, the surface of the sphere. Then,

$$S = 8T; \quad \therefore \quad L : 8T :: A : 4; \quad \therefore \quad L = T \times 2A.$$

3. *An ungula is equivalent to the product of a tri-rectangular spherical pyramid by twice the angle of the ungula, the right angle being the unit.* (?)

477. Proposition XXVII.—Theorem.

If two semi-circumferences of great circles intersect on the surface of a hemisphere, the sum of the opposite triangles thus formed is equivalent to a lune whose angle is equal to that included by the semi-circumferences.



Let the semi-circumferences, BAD, CAE, intersect at A on the surface of a hemisphere; then will the sum of the opposite triangles, BAC, DAE, be equivalent to a lune whose angle is BAC.

For, the semi-circumferences produced around the sphere intersect on the opposite hemisphere at A'.

All great circles bisect each other; hence, each of the arcs, AD, $A'B$, is the supplement of AB; $\therefore$ $AD = A'B$.

Likewise, $AE = A'C$, $DE = BC$; hence, the triangles, ADE, $A'BC$, are symmetrical, and therefore equivalent; hence, $ABC + ADE = ABC + A'BC = ABA'CA$, a lune whose angle is BAC.

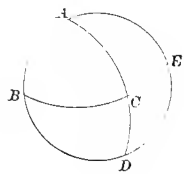
478. Corollary.

The sum of two spherical pyramids, the sum of whose bases is equivalent to a lune, is equivalent to an ungula whose base is the lune. (?)

479. Proposition XXVIII.—Theorem.

The area of a spherical triangle is equal to its spherical excess multiplied by the area of the tri-rectangular triangle.

Let ABC be a spherical triangle. Complete the circumference $ABDE$; produce AC, BC, to meet this circumference in D, E.



$$ABC + BCD = \text{lune } A = T \times 2A,$$

$$ABC + ACE = \text{lune } B = T \times 2B,$$

$$ABC + DCE = \text{lune } C = T \times 2C.$$

$$2ABC + ABC + BCD + ACE + DCE = T \times 2(A + B + C).$$

But, $\quad ABC + BCD + ACE + DCE = 4T.$ (?)

$$\therefore \quad 2ABC + 4T = T \times 2(A + B + C).$$

$$\therefore \quad ABC = T(A + B + C - 2).$$

Thus, if $A = 100°$, $B = 120°$, $C = 140°$, a right angle being the unit,

$$ABC = T\left(\frac{100°}{90°} + \frac{120°}{90°} + \frac{140°}{90°} - 2\right) = 2T.$$

480. Corollaries.

1. *The volume of a triangular spherical pyramid is equal to the spherical excess of its base multiplied by the area of the tri-rectangular spherical pyramid.* (?)

2. *Two triangular spherical pyramids, on the same sphere or on equal spheres, are proportional to their bases.* (?)

481. Proposition XXIX.—Theorem.

The area of a spherical polygon is equal to its spherical excess multiplied by the area of the tri-rectangular triangle.

Let P denote the area of a spherical polygon; S, the sum of its angles; n, the number of its sides; t, t', t'', ... the areas of the triangles formed by drawing diagonals from any vertex; s, s', s'', ... respectively, the sums of the angles of these triangles; T, the area of the tri-rectangular triangle.

Then, $t = (s - 2)T, \quad t' = (s' - 2)T, \quad t'' = (s'' - 2)T, \ldots$

$\therefore \quad t + t' + t'' + \ldots = [s + s' + s'' + \ldots - 2(n - 2)]T.$

But, $t + t' + t'' + \ldots = P, \quad s + s' + s'' + \ldots = S.$

$\therefore \quad P = [S - 2\,(n - 2)]T.$

482. Corollaries.

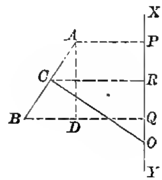
1. *Any two spherical pyramids, on the same sphere or on equal spheres, are proportional to their bases.* (?)

2. *Any spherical pyramid is to the sphere as its base is to the surface of the sphere.* (?)

483. Proposition XXX.—Lemma.

The area of the surface generated by the revolution of a straight line about an axis in the same plane, is equal to the

product of the projection of the line on the axis by the circumference whose radius is the perpendicular to the line erected at its middle point and terminated by the axis.



Let the straight line AB revolve about the axis XY in the same plane, and let PQ be its projection on the axis; CO, the perpendicular to AB at its middle point C, terminating in the axis. Then,

$$\text{area } AB = PQ \times 2\pi . OC.$$

The surface generated by AB is the lateral surface of a frustum of a cone of revolution; hence, drawing CR perpendicular and AD parallel to XY,

$$(1) \quad \text{area } AB = AB \times 2\pi . CR \quad (\mathbf{440},\ 4).$$

The triangles, ABD, COR, are similar; (?)

$$\therefore \qquad AD : AB :: CR : OC.$$

$$\text{But, } AD = PQ, \qquad CR : OC :: 2\pi . CR : 2\pi . OC.$$

$$\therefore \qquad PQ : AB :: 2\pi . CR : 2\pi . OC.$$

$$\therefore \qquad PQ \times 2\pi . OC = AB \times 2\pi . CR.$$

$$\therefore \quad (2) \qquad \text{area } AB = PQ \times 2\pi . OC.$$

If either extremity of AB is in the axis XY, AB generates the lateral surface of a cone of revolution, and formula (2) still holds true. (?)

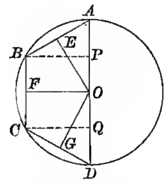
If AB is parallel to the axis XY, it generates the lateral surface of a cylinder of revolution, and formula (2) still holds true. (?)

If AB is perpendicular to the axis XY, it generates a circular ring; PQ becomes zero; CO becomes infinite; formula (2) gives an indeterminate result, and therefore fails; but formula (1) holds true. (?)

484. Proposition XXXI.—Theorem.

The area of the surface of a sphere is equal to the product of its diameter by the circumference of a great circle.

Let the semicircle ABD, and any regular inscribed semi-polygon, as $ABCD$, revolve together about the diameter AD.



The semi-circumference will generate the surface of a sphere, and the semi-perimeter, a surface equal to the sum of the surfaces generated by its sides.

OE, OF, OG, drawn from the center to the middle points of the chords, AB, BC, CD, are perpendicular to the chords and equal to each other. (?)

$\therefore$ area $AB = AP \times 2\pi . OE$; (?)

also, area $BC = PQ \times 2\pi . OE$; (?)

and, area $CD = QD \times 2\pi . OE$; (?)

$\therefore$ area $ABCD = (AP + PQ + QD) \times 2\pi . OE$.

$\therefore$ area $ABCD = AD \times 2\pi . OE$.

Now, if the number of sides of the regular inscribed semi-polygon be indefinitely increased, the surface generated by the semi-perimeter will approach the surface of the sphere as its limit, and OE will approach OA.

$\therefore$ area of surface of sphere $= AD \times 2\pi . OA$ (**301**).

If s denote the surface, d the diameter, c the circumference,

$$s = d \times c.$$

485. Corollaries.

1. *The surface of a sphere is equivalent to four great circles.*

For, if r denote the radius of the sphere,

$$s = d \times c = 2r \times 2\pi r = 4\pi r^2.$$

2. *The surfaces of two spheres are proportional to the squares of their radii.*

For, $s : s' :: 4\pi r^2 : 4\pi r'^2 :: r^2 : r'^2$.

3. *The surface of a sphere is equivalent to a circle whose radius is equal to the diameter of the sphere.*

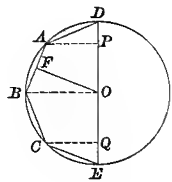
For, $s = 4\pi r^2 = \pi(2r)^2 = \pi d^2$.

4. *The surfaces of two spheres are proportional to the squares of their diameters.*

For, $s : s' :: \pi d^2 : \pi d'^2 :: d^2 : d'^2$.

486. Proposition XXXII.—Theorem.

The area of a zone is equal to the product of its altitude by the circumference of a great circle.



If the semicircle DBE, the arc AC, together with the equal chords, AB, BC, revolve about the diameter DE as an axis, the semi-circumference will generate the surface of a sphere, the arc AC a zone, and the chords, AB, BC, a surface whose area is expressed by the formula,

$$\text{area } ABC = PQ \times 2\pi . OF. \quad (?)$$

If the number of equal arcs into which AC is divided be increased indefinitely, the surface generated by the chords of these arcs will approach the zone as its limit, the perpendicular OF will approach the radius of the sphere as its limit, and PQ will remain constant.

$$\therefore \quad \text{area zone } AC = PQ \times 2\pi . OD. \quad (?)$$

Denoting the area of the zone by z, its altitude by a, and the radius of the sphere by r, the above formula becomes

$$z = 2\pi ra.$$

487. Corollaries.

1. *Zones on the same sphere or on equal spheres are proportional to their altitudes.* (?)

2. *A zone is to the surface of the sphere as the altitude of the zone is to the diameter of the sphere.* (?)

3. *A zone of one base is equivalent to the circle whose radius is the chord of the generating arc.*

For, zone $DA = DP \times 2\pi . OD$;

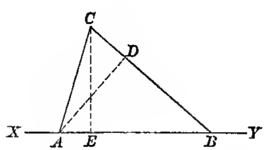
$\therefore$ zone $DA = \pi . DP \times DE = \pi . \overline{DA}^2$ (**240**, 3, 2*d*).

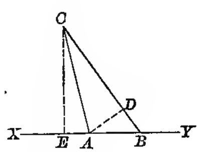
4. If a becomes $2r$, z becomes s, the surface of the sphere. Then, $z = 2\pi ra$ becomes $s = 4\pi r^2$.

488. Proposition XXXIII.—Lemma.

The volume generated by the revolution of a plane triangle about an exterior axis, in the plane of the triangle and through its vertex, is equal to the product of the area generated by the base by one-third of the altitude of the triangle.

1. When one side of the triangle lies in the axis:





Let the side AB of the triangle ABC lie in the axis XY. Draw AD perpendicular to BC, and CE perpendicular to XY.

The volume generated by ABC is the sum or the difference of the volumes of the cones generated by the right

triangles, BEC, AEC, according as CE is within or without the triangle.

But, $\quad$ vol. $BEC = \frac{1}{3}\pi.\overline{EC}^2 \times BE$, (?)

and, $\quad$ vol. $AEC = \frac{1}{3}\pi.\overline{EC}^2 \times AE$; (?)

$\therefore$ vol. $BEC \pm$ vol. $AEC = \frac{1}{3}\pi.\overline{EC}^2 \times (BE \pm AE)$.

Taking the plus signs for the first figure, and observing that $BE + AE = AB$, and taking the minus signs for the second figure, and observing that $BE - AE = AB$, we have

$$\text{vol. } ABC = \tfrac{1}{3}\pi.\overline{EC}^2 \times AB = \tfrac{1}{3}\pi.EC \times EC \times AB.$$

But $EC \times AB = BC \times AD$, since each is twice the area of the triangle;

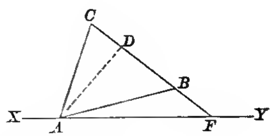
$$\therefore \quad \text{vol. } ABC = \tfrac{1}{3}\pi.EC \times BC \times AD.$$

But $\pi.EC \times BC =$ area of surface generated by BC (**439**);

$$\therefore \quad \text{vol. } ABC = \text{area } BC \times \tfrac{1}{3}AD.$$

2. When only the vertex is in the axis, and the base not parallel to the axis:

In this case, the volume generated by ABC is the difference of the volumes generated by the triangles, ACF, ABF.



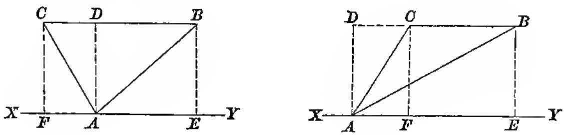
But, $\quad$ vol. $ACF =$ area $CF \times \frac{1}{3}AD$,

and $\quad$ vol. $ABF =$ area $BF \times \frac{1}{3}AD$.

$\therefore$ vol. $ACF -$ vol. $ABF =$(area $CF -$ area BF) $\times \frac{1}{3}AD$.

$\therefore \quad$ vol. $ABC =$ area $BC \times \frac{1}{3}AD$.

3. When only the vertex is in the axis, and the base parallel to the axis:



In this case, the volume generated by the triangle ABC is the sum or the difference of the volumes generated by the right triangles, ADB, ADC, according as the perpendicular AD is within or without the triangle.

Drawing BE, CF, perpendicular to the axis, the volume generated by the right triangle ADB is the volume of the cylinder generated by the rectangle $ADBE$, minus the volume of the cone generated by AEB.

But, vol. $ADBE = \pi . \overline{AD}^2 \times BD$, (?)

and vol. $AEB = \frac{1}{3}\pi . \overline{AD}^2 \times BD$. (?)

$\therefore$ vol. $ADBE$ — vol. $AEB = \frac{2}{3}\pi . \overline{AD}^2 \times BD$.

$\frac{2}{3}\pi . \overline{AD}^2 \times BD = 2\pi . AD \times BD \times \frac{1}{3}AD =$ area $BD \times \frac{1}{3}AD$.

$\therefore$ vol. $ADB =$ area $BD \times \frac{1}{3}AD$.

Likewise, vol. $ADC =$ area $CD \times \frac{1}{3}AD$.

$\therefore$ vol. $ADB \pm$ vol. $ADC =$ area $(BD \pm CD) \times \frac{1}{3}AD$.

Taking the plus signs for the first figure, and observing that area $(BD + CD) =$ area BC, and taking the minus signs for the second figure, and observing that area $(BD - CD)$ $=$ area BC, and that the first member is equal to vol. ABC, we have

vol. $ABC =$ area $BC \times \frac{1}{3}AD$.

489. Corollary.

In each of the three preceding cases, if the triangle ABC is isosceles, AD is the perpendicular to BC erected at its middle point and terminated by the axis.

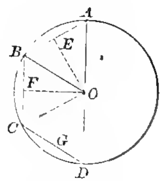
Denoting this perpendicular by p, and the projection of BC on the axis XY by a, we have

$$\text{area } BC = a \times 2\pi p = 2\pi pa. \quad (\mathbf{483}).$$

$$\therefore \quad \text{vol. } ABC = 2\pi pa \times \tfrac{1}{3}p = \tfrac{2}{3}\pi p^2 a.$$

490. Proposition XXXIV.—Theorem.

The volume of a sphere is equal to the product of the area of its surface by one-third of its radius.



Let the semicircle ABD, and any regular inscribed semi-polygon, as $ABCD$, with radii drawn to the vertices, revolve together about the diameter AD as an axis.

The semicircle will generate a sphere, and the semi-polygon, a volume equal to the sum of the volumes generated by the triangles, AOB, BOC, COD.

OE, OF, OG, drawn from the center to the middle points of the chords, AB, BC, CD, are perpendicular to the chords and equal to each other. (?)

$\therefore$ vol. AOB = area $AB \times \frac{1}{3}OE$, (?)

also, vol. BOC = area $BC \times \frac{1}{3}OE$, (?)

and vol. COD = area $CD \times \frac{1}{3}OE$. (?)

$\therefore$ vol. $ABCD$ = area $ABCD \times \frac{1}{3}OE$.

Whatever be the number of sides of the regular semi-polygon, the volume generated is equal to the product of

the area of the surface generated by the semi-perimeter by one-third of the apothem.

Now, if the number of sides be increased indefinitely, the volume generated will approach the volume of the sphere as its limit, the surface generated will approach the surface of the sphere, and the apothem will approach the radius.

$\therefore$ vol. of sphere $=$ area of surface $\times$ $\frac{1}{3}$ radius (**303**).

Denoting the volume of the sphere by v, the area of its surface by s, the radius by r, we have

$$v = s \times \tfrac{1}{3}r.$$

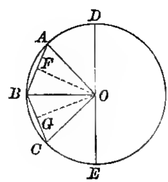
491. Corollaries.

1. (1) $v = \frac{4}{3}\pi r^3$; (?) (2) $v = \frac{1}{6}\pi d^3$. (?)

2. *The volumes of two spheres are proportional to the cubes of their radii or to the cubes of their diameters.* (?)

492. Proposition XXXV.—Theorem.

The volume of a spherical sector is equal to the product of the area of the zone which forms its base by one-third of the radius of the sphere.



If the semicircle DBE, the arc AC, and the equal triangles, AOB, BOC, revolve about the diameter DE as an axis, the triangles will generate a volume expressed by the formula,

$$\text{vol. } OABC = \text{area } ABC \times \tfrac{1}{3}OF. \quad (?)$$

Whatever be the number of equal triangles, the volume generated will be equal to the area of the surface generated

by their bases, multiplied by one-third of their common altitude.

Now, if the number of equal triangles be increased indefinitely, the volume generated will approach the spherical sector OAC as its limit, the surface generated by the bases will approach the zone generated by the arc AC, and the common altitude will approach the radius.

$\therefore$ vol. spherical sector $OAC =$ zone $AC \times \frac{1}{3}$ radius (**303**).

Denoting the volume of the spherical sector by v, the area of the zone which forms its base by z, the radius by r, we have

$$v = z \times \tfrac{1}{3}r.$$

493. Corollaries.

1. If a denotes the altitude of the zone, $z = 2\pi ra$;

$$\therefore \quad v = \tfrac{2}{3}\pi r^2 a.$$

2. *Two spherical sectors of the same sphere are proportional to the altitudes of the zones which form their bases.* (?)

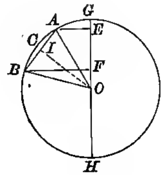
3. If a becomes $2r$; z becomes s, the surface of the sphere; and v becomes the volume of the sphere;

$$\therefore \quad v = s \times \tfrac{1}{3}r, \quad \text{or} \quad v = \tfrac{4}{3}\pi r^3.$$

494. Proposition XXXVI.—Theorem.

The solid generated by the revolution of a circular segment about an exterior diameter as an axis, is equivalent to one-sixth of the cylinder whose radius is equal to the chord of the segment, and whose altitude is equal to the projection of the chord on the axis.

Let the circular segment ABC revolve about the exterior diameter GH as an axis;



$v =$ vol. generated by ABC;

$v' =$ vol. generated by $ACBO$;

$v'' =$ vol. generated by ABO;

$a = EF, \quad c = AB, \quad p = OI,$

the perpendicular to AB from the center;

$$v = v' - v'';$$

$$v' = \tfrac{2}{3}\pi r^2 a \quad (\mathbf{493}, 1); \quad \text{and} \quad v'' = \tfrac{2}{3}\pi p^2 a \quad (\mathbf{489});$$

$$v' - v'' = \tfrac{2}{3}\pi (r^2 - p^2) a; \quad \text{but,} \quad r^2 - p^2 = \tfrac{1}{4}c^2; \quad (?)$$

$$\therefore \quad v = \tfrac{1}{6}\pi c^2 a.$$

495. Corollaries.

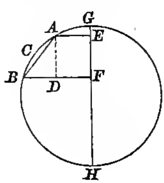
1. The solid generated by the circular segment whose altitude is a and chord c, is to the sphere whose diameter is c as $a : c$. (?)

2. *The solid generated by a circular segment whose chord is parallel to the axis, is equivalent to a sphere whose diameter is equal to the chord.* (?)

496. Proposition XXXVII.—Theorem.

The volume of a spherical segment is equal to the product of one-half of its altitude by the sum of its bases, plus the volume of a sphere whose diameter is the altitude of the segment.

Let v denote the volume; a, the altitude EF; c, the chord AB; r, r', the radii, BF, AE, of the bases of the spherical segment generated by the revolution of $ACBFE$ about the diameter GH.



Denoting the volume generated by the circular segment ABC by v', and the frustum of a cone generated by the trapezoid $ABFE$ by v'', we have $v = v' + v''$;

$$v' = \tfrac{1}{6}\pi c^2 a; \qquad v'' = \tfrac{1}{3}\pi a(r^2 + r'^2 + rr') \quad (\mathbf{443,}\ 5);$$

$$c^2 = (r - r')^2 + a^2 = r^2 + r'^2 - 2rr' + a^2;$$

$$\therefore \quad v' = \tfrac{1}{6}\pi a(r^2 + r'^2 - 2rr') + \tfrac{1}{6}\pi a^3;$$

$$\therefore \quad v = \tfrac{1}{2}a(\pi r^2 + \pi r'^2) + \tfrac{1}{6}\pi a^3. \quad (?)$$

497. Corollaries.

1. If A coincides with G, then $r' = 0$, the segment has only one base, and $v = \frac{1}{2}a\pi r^2 + \frac{1}{6}\pi a^3$.

2. If A coincides with G, and B with H, then $r' = 0$, $r = 0$, $a = d$, the diameter of the sphere, the segment becomes the sphere, and $v = \frac{1}{6}\pi d^3$.

498. Formulas for the Sphere.

1. $r = \frac{1}{2}d.$	11. $c = \sqrt{\pi s}.$
2. $r = \dfrac{c}{2\pi}\cdot$	12. $c = \sqrt[3]{6v\pi^2}.$
3. $r = \frac{1}{2}\sqrt{\dfrac{s}{\pi}}\cdot$	13. $s = 4\pi r^2$
4. $r = \frac{1}{2}\sqrt[3]{\dfrac{6v}{\pi}}\cdot$	14. $s = \pi d^2.$
5. $d = 2r.$	15. $s = \dfrac{c^2}{\pi}\cdot$
6. $d = \dfrac{c}{\pi}\cdot$	16. $s = \sqrt[3]{36\pi v^2}.$
7. $d = \sqrt{\dfrac{s}{\pi}}\cdot$	17. $v = \frac{4}{3}\pi r^3.$
8. $d = \sqrt[3]{\dfrac{6v}{\pi}}\cdot$	18. $v = \frac{1}{6}\pi d^3.$
9. $c = 2\pi r.$	19. $v = \dfrac{c^3}{6\pi^2}\cdot$
10. $c = \pi d.$	20. $v = \dfrac{s}{6}\sqrt{\dfrac{s}{\pi}}\cdot$

499. Exercises.

1. If L denotes the area of a lune, A its angle, r the radius of the sphere, prove that $L = A\pi r^2$.

2. If P denotes the area of a spherical polygon, e its spherical excess, r the radius of the sphere, prove that $P = \frac{1}{2}\pi r^2 e$.

3. The volume of any spherical pyramid is equal to the product of the area of its base by one-third of the radius of the sphere.

4. If v denotes the volume of a spherical pyramid, e the spherical excess of its base, r the radius of the sphere, prove that $v = \frac{1}{6}\pi r^3 e$.

5. A sphere 6 in. in diameter is bored through the center with a 3-inch auger; required the volume remaining.

6. Find the ratio of the surfaces of a sphere and the circumscribed cube, also the ratio of their volumes.

7. Find the ratio of the surfaces of a sphere and the circumscribed cylinder, also the ratio of their volumes.

8. The diameter of a hollow sphere of glass is 4 ft., the thickness of the shell is 1 in. If this sphere be melted and run into a solid sphere, what would be its diameter?

9. The volume of a cone is 1,000 cu. in., the radius of its base is 6 in.; required the volume of the circumscribed sphere.

10. The surface of a sphere is s; what is the surface of a sphere whose volume is n times as great?

11. The volume of a sphere is v; what is the volume of a sphere whose surface is n times as great?

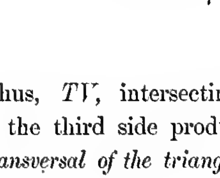
BOOK VIII.

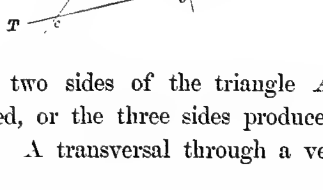
MODERN GEOMETRY.

I. TRANSVERSALS.

500. Definitions.

1. A **transversal** is a straight line intersecting a system of lines.





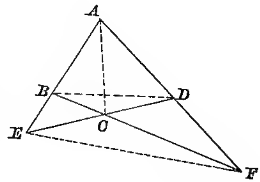
Thus, TV, intersecting two sides of the triangle ABC and the third side produced, or the three sides produced, is a *transversal of the triangle*. A transversal through a vertex is an *angle-transversal*.

2. The **segments** of the sides are the distances of the points of intersection of the transversal and sides from the vertices. Thus, Ac, Bc; Ab, Cb; Ba, Ca.

3. **Adjacent segments** are segments having a common extremity. Thus, Ac, Bc; Ab, Cb; Ba, Ca; Ab, Ac; Ba, Bc; Ca, Cb.

4. **Non-adjacent segments** are segments not having a common extremity. Thus, Ac, Ba, Cb; Ab, Bc, Ca.

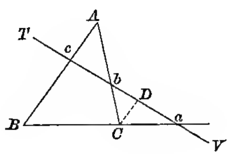
5. A **complete quadrilateral** is the figure formed by four straight lines intersecting in six points.

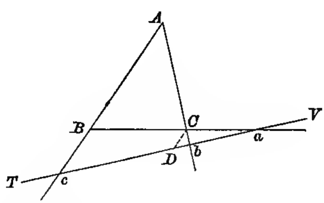


Thus, $ABCDEF$ is a complete quadrilateral; AC, BD, EF, are its diagonals, the external diagonal EF being called the third diagonal.

501. Proposition I.—Theorem.

A transversal of a triangle divides the sides into segments such that the product of three non-adjacent segments is equal to the product of the other three segments.





Let TV be a transversal of the triangle ABC. Draw CD parallel to AB; then, from similar triangles,

$$Ba : Ca :: Bc : CD, \quad Cb : Ab :: CD : Ac;$$

$$\therefore \quad Ba \times Cb : Ab \times Ca :: Bc : Ac;$$

$$\therefore \quad Ac \times Ba \times Cb = Ab \times Bc \times Ca.$$

502. Corollary.

Three points in the sides of a triangle, one being in a side produced, or all three being in the sides produced, dividing the sides so that the product of three non-adjacent segments is equal to the product of the other three segments, are in the same straight line.

Let a, b, c, be points such that

$$(1) \quad Ac \times Ba \times Cb = Ab \times Bc \times Ca.$$

Let the straight line ab cut the third side in c'; then,

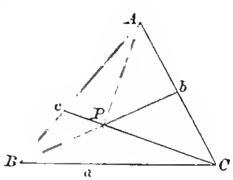
$$(2) \quad Ac' \times Ba \times Cb = Ab \times Bc' \times Ca \quad (\mathbf{501}).$$

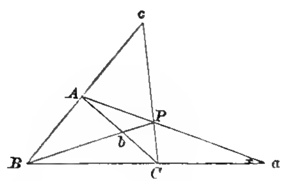
$(1) \div (2) = (3) \quad Ac : Ac' :: Bc : Bc'.$

This is true only when c, c', coincide. (ALG., **317,** 3; **116,** 1.) But abc' is a straight line; $\therefore$ abc is a straight line.

503. Proposition II.—Theorem.

The three angle-transversals of a triangle through a common point divide the sides so that the product of three non-adjacent segments is equal to the product of the other three segments.





Let Aa, Bb, Cc, through P be angle-transversals of ABC. The transversals, Bb, Cc, of the triangles, AaC, ABa, give

$$AP \times aB \times Cb = Ab \times aP \times CB;$$

$$Ac \times BC \times aP = AP \times Bc \times aC;$$

$$\therefore \quad Ac \times Ba \times Cb = Ab \times Bc \times Ca.$$

504. Corollaries.

1. *Three angle-transversals of a triangle dividing the sides so that the product of three non-adjacent segments is equal to the product of the other three segments, pass through a common point.* (?)

2. *If one of three angle-transversals of a triangle, through the same point, bisects one side, the line joining the points of division of the other sides is parallel to that side.* (?)

3. *If three angle-transversals of a triangle pass through the same point, and the line joining the points of division of two sides is parallel to the third side, this side is bisected by the other point.* (?)

505. Exercises.

1. The three medial lines of a triangle pass through the same point (**51,** 17), (**504,** 1).

2. The three bisectors of the angle of a triangle pass through the same point (**221**), (**504,** 1).

3. The three altitudes of a triangle pass through the same point (**227,** 1), (**504,** 1).

4. The three angle-transversals of a triangle to the points of tangency of the inscribed circle pass through the same point (**176,** 1), (**504,** 1).

5. The transversal through the intersection of the non-parallel sides of a trapezoid and the intersection of the diagonals bisects the parallel sides (**503**), (**217**).

6. The middle points of the three diagonals of a complete quadrilateral are in the same straight line (**501**), (**502**).

II. HARMONIC PROPORTION.

506. Definitions.

1. Three quantities are in *harmonic proportion* if the difference of the first and second is to the difference of the second and third as the first is to the third; and the second is the *harmonic mean* between the first and third.

2. A **pencil** is a system of lines diverging from a point.

3. A **ray** is one of the lines of a pencil.

4. The **vertex** of a pencil is the point from which the rays diverge.

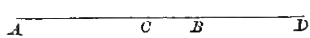
5. A **harmonic pencil** is a pencil of four rays dividing a transversal harmonically (**250,** 3, 4, 5, 6).

6. **Conjugate rays** are the alternate rays of a harmonic pencil.

7. Two circles intersect each other *orthogonally* if the tangents at the common point are at right angles.

507. Proposition III.—Theorem.

The distance between the points of division of a line divided harmonically is the harmonic mean between the distances from the extremities of the line to the point of division not between them.



Let C, D, divide AB harmonically. Then,

$$AC : BC :: AD : BD.$$

That is, $AD - CD : CD - BD :: AD : BD.$

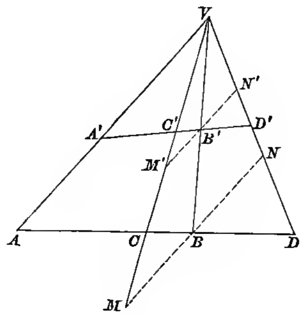
$\therefore$ CD is the harmonic mean between AD and BD.

508. Corollary.

A line divided harmonically is the harmonic mean between the distances from the extremity not between the points of division to the points of division (**250**, 4), (**507**).

509. Proposition IV.—Theorem.

Every transversal of a harmonic pencil is divided harmonically at the points of intersection.



If the pencil $V\text{-}ACBD$ divides AB harmonically, it will divide any other transversal $A'B'$ harmonically. Through B, B', draw MN, $M'N'$, parallel to AV.

$\therefore\ AC : BC :: AV : BM,$

$AV : BN :: AD : BD.$

By hypothesis, we have

$$AC : BC :: AD : BD; \quad \therefore\ AV : BM :: AV : BN;$$

$$\therefore\ BM = BN; \quad \therefore\ B'M' = B'N' \text{ (**237**)}.$$

Similar triangles give the proportions,

$$A'C' : B'C' :: A'V : B'M', \quad A'D' : B'D' :: A'V : B'N'.$$

Since $\quad B'M' = B'N', \quad A'C' : B'C' :: A'D' : B'D'.$

Hence, $A'B'$ is divided harmonically at C', D'.

510. Exercises.

1. If the straight line AB is divided internally at C and externally at D, so that CD is the harmonic mean between AD and BD, then $AC \times BD = BC \times AD$ (**506**, 1).

2. If the straight line AD is divided by the points B, C, so that $AC \times BD = BC \times AD$, then AB is divided harmonically at the points C, D, and CD at the points A, B.

3. Any diagonal of a complete quadrilateral is divided harmonically by the other two.

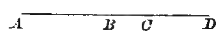
4. If two circles cut each other orthogonally, any diameter of either intersecting the circumference of the other is divided harmonically by that circumference, and conversely (**248**).

III. ANHARMONIC RATIO.

511. Definitions.

1. The **anharmonic ratio of four points in a straight line** is the ratio of the rectangle of the distances between the first and third and the second and fourth to the rectangle of the distances between the first and fourth and the second and third.

Thus, $AC \times BD : AD \times BC$ is the anharmonic ratio of the points A, B, C, D, which is briefly expressed by writing the letters in order in brackets, thus, $[ABCD] = AC \times BD \div AD \times BC$.



2. The **anharmonic ratio of a pencil of four rays** is the anharmonic ratio of the four points of intersection of these rays by any transversal.

3. The **anharmonic ratio of four points in the circumference of a circle** is the anharmonic ratio of the pencil formed by joining these points with any point in the circumference.

4. The **anharmonic ratio of four tangents** is the anharmonic ratio of the points of intersection of the four tangents by any fifth tangent.

512. Proposition V.—Theorem.

The anharmonic ratio of four points is not changed by interchanging two of the letters designating the points, provided the other two are also interchanged.

$$[ABCD] = AC \times BD : AD \times BC.$$
$$[BADC] = BD \times AC : BC \times AD.$$
$$[CDAB] = CA \times DB : CB \times DA.$$
$$[DCBA] = DB \times CA : DA \times CB.$$
$$\therefore\ [ABCD] = [BADC] = [CDAB] = [DCBA].$$

513. Scholiums.

1. Since four letters taken all in a set have twenty-four permutations, four points have six different anharmonic ratios, each having four expressions.

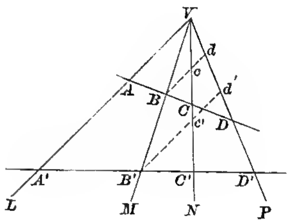
2. $[ABCD] = AC \times BD : AD \times BC = \frac{AC}{BC} : \frac{AD}{BD}$.

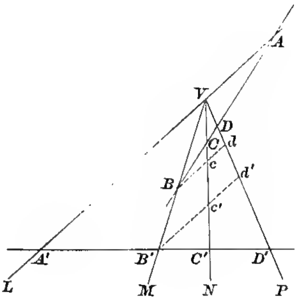
3. If four points, A, B, C, D, are harmonic, $\frac{AC}{BC} = \frac{AD}{BD}$, and $\frac{AC}{BC} : \frac{AD}{BD} = 1$. But four points taken at random in a straight line are not, in general, harmonic, $\frac{AC}{BC}$ is not equal to $\frac{AD}{BD}$, and $\frac{AC}{BC} : \frac{AD}{BD}$ is anharmonic.

514. Proposition VI.—Theorem.

Any anharmonic ratio of the points of intersection of a pencil of four rays by a variable transversal is constant.

Let $ABCD$, $A'B'C'D'$, be any two positions of a variable transversal of the pencil $V\text{-}LMNP$.





Draw Bd, $B'd'$, parallel to AV; then, by similar triangles,

$$(1)\quad \frac{AV}{Bc} = \frac{AC}{BC}, \qquad (2)\quad \frac{AV}{Bd} = \frac{AD}{BD},$$

$$(3)\quad \frac{A'V}{B'c'} = \frac{A'C'}{B'C'}, \qquad (4)\quad \frac{A'V}{B'd'} = \frac{A'D'}{B'D'}.$$

$$(1) \div (2) = (5)\quad \frac{Bd}{Bc} = \frac{AC \times BD}{AD \times BC} = [ABCD].$$

$$(3) \div (4) = (6)\quad \frac{B'd'}{B'c'} = \frac{A'C' \times B'D'}{A'D' \times B'C'} = [A'B'C'D'].$$

But, $\dfrac{Bd}{Bc} = \dfrac{B'd'}{B'c'}$ (**237**); $\therefore$ $[ABCD] = [A'B'C'D']$.

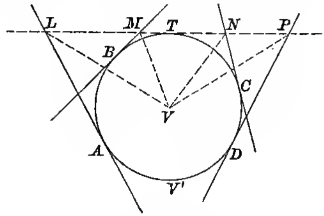
515. Corollaries.

1. *If the angles of two pencils are respectively equal, their anharmonic ratios are equal.*

2. *The anharmonic ratio of four fixed points in the circumference of a circle is constant* (**511,** 3), (**154**), (**515,** 1).

516. Proposition VII.—Theorem.

The anharmonic ratio of four fixed tangents to a circle is constant.



Let A, B, C, D, be the points of tangency of the fixed tangents; L, M, N, P, the intersections of these tangents by a fifth tangent. Then, by **176,** 3,

LVM is measured by $\frac{1}{2}AT - \frac{1}{2}BT = \frac{1}{2}AB$, (?)
MVN is measured by $\frac{1}{2}BT + \frac{1}{2}TC = \frac{1}{2}BC$, (?)
NVP is measured by $\frac{1}{2}DT - \frac{1}{2}CT = \frac{1}{2}DC$. (?)

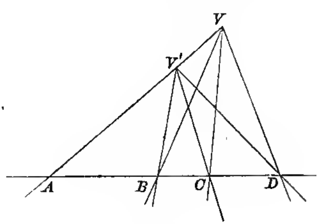
Hence, the angles of the pencil V-$LMNP$ are constant; therefore, the anharmonic ratio $[LMNP]$ is constant (**515,** 1).

517. Corollary.

The anharmonic ratio of four tangents to a circle is equal to the anharmonic ratio of the four points of tangency. (?)

518. Proposition VIII.—Theorem.

If two pencils have the same anharmonic ratio and a common homologous ray, the intersections of the other homologous rays are in the same straight line.



Let the pencils V-$ABCD$, V'-$ABCD$, have the same anharmonic ratio and a common homologous ray, VA. Let the transversal through B, C, two points of intersection of homologous rays, intersect the ray VA in A, the ray VD in D', the ray $V'D$

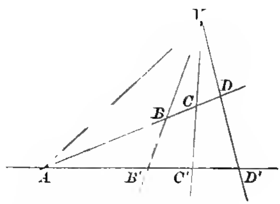
in D''. Then, $[ABCD'] = [ABCD'']$; (?) hence, D, D', are coincident; (?) and, since D' is in VD, D'' in $V'D$, both D', D'', must be at D, the intersection of VD, $V'D$. Hence, the points B, C, D, are in the same straight line.

519. Corollary.

If one anharmonic ratio of a pencil is equal to one anharmonic ratio of another pencil, the other anharmonic ratios of the first are respectively equal to those of the second. (?)

520. Proposition IX.—Theorem.

If two straight lines have one anharmonic ratio of four points of the one equal to one anharmonic ratio of four points of the other, and two homologous points coincident, the straight lines through the other three pairs of homologous points meet in a common point.



Let $[ABCD] = [AB'C'D']$; A, the coincident points; V, the intersection of $B'B$ and $C'C$. Draw VA, VD', and denote the point in which VD' cuts AB by D''.

$$\text{Then,} \qquad [ABCD''] = [AB'C'D'] \quad (\mathbf{514}).$$

$$\therefore \qquad [ABCD''] = [ABCD].$$

Hence, D'', D, coincide; (?) $\therefore$ the straight line $D'D$ passes through V.

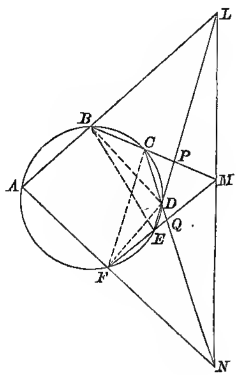
521. Corollary.

If one anharmonic ratio of a system of four points is equal to one anharmonic ratio of another system, the other anharmonic

ratios of the first system are respectively equal to those of the second. (?)

522. Proposition X.—Theorem.

The intersections of the three pairs of opposite sides of an inscribed hexagon lie in the same straight line.



The pencils formed by drawing rays from B, F, to A, C, D, E, have the same anharmonic ratio (**515**, 2).

The first is cut by the transversal $LPDE$, the second by $NCDQ$; $\therefore$ $[LPDE]=[NCDQ]$; $\therefore$ LN, PC, EQ, meet in the same point M (**520**); $\therefore$ L, M, N, the intersections of the three pairs of opposite sides of the inscribed hexagon, lie in the same straight line.

523. Corollaries.

1. *The intersection of one side of an inscribed pentagon with the tangent at the opposite vertex, and the intersection of the other non-consecutive sides, are three points in the same straight line.*

For, as the vertex D of the last figure approaches C, the side CD approaches the tangent at C; but the theorem for the hexagon holds for all positions of D up to C, when the hexagon becomes a pentagon and CD a tangent.

2. *If tangents be drawn at two consecutive vertices of an inscribed quadrilateral, the point of intersection of each with the*

side through the point of tangency of the other, and the intersection of the other two sides, are three points in the same straight line.

This becomes evident by supposing D in the hexagon to move to C, and E to F.

3. *The intersections of the tangents drawn at the opposite vertices of an inscribed quadrilateral, and the intersections of the pairs of opposite sides, are four points in the same straight line.*

For, if in the inscribed hexagon F move to A, and D to C, the intersection of the tangents at the opposite vertices, A, C, will be on the straight line through the intersections of the opposite sides of the quadrilateral $ABCE$; but the same quadrilateral can be obtained from another hexagon so as to form tangents at the other opposite vertices, B, E; but the intersection of these tangents is on the line through the intersections of the opposite sides; hence, the four points of intersection are in the same straight line.

4. *The intersections of the sides of an inscribed triangle with the tangents at the opposite vertices lie in the same straight line.*

This becomes evident by supposing F in the hexagon to move to A, C to B, E to D.

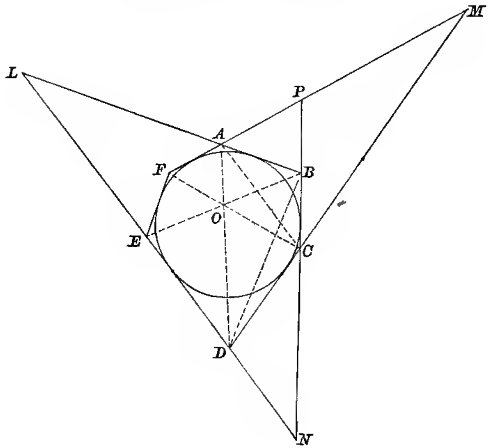
REMARK.—Theorem X is due to *Pascal*, and Theorem XI to *Brianchon*. These theorems with their corollaries afford a beautiful exemplification of the method of anharmonic ratio, and will well repay the student for the labor which their mastery will require.

524. Proposition XI.—Theorem.

The three diagonals joining the opposite vertices of a circumscribed hexagon pass through the same point.

Regard AB, BC, CD, EF, as fixed tangents, cut by the tangent LN in L, N, D, E, and by the tangent FM in A, P, M, F; then, (**516**),

$$[LNDE] = [APMF].$$



Hence, the pencils, B-$LNDE$, C-$APMF$, have equal anharmonic ratios and a common homologous ray PN; therefore the intersections, A, D, O, of BL, CA; BD, CM; BE, CF, lie in the same straight line (**518**); hence, the diagonals, AD, BE, CF, pass through the same point.

525. Corollaries.

1. *The line joining a vertex of a circumscribed pentagon and the point of tangency of the opposite side, and the diagonals drawn from the extremities of this side to the extremities of the adjacent sides, meet in the same point.*

This becomes evident by moving C to the circumference.

2. *The lines joining the points of tangency of each of two*

adjacent sides of a circumscribed quadrilateral with the vertex at the extremity of the other, and the diagonal joining the other vertices, meet in the same point.

This becomes evident by moving C and E to the circumference.

3. *The diagonals of a circumscribed quadrilateral and the lines joining the points of tangency of the opposite sides meet at the same point.*

For, if C and F move to the circumference, the line joining the points of tangency of two opposite sides passes through the intersection of the diagonals of the quadrilateral; but the same quadrilateral can be obtained from another hexagon by moving two vertices to the points of tangency of the other opposite sides, the diagonal joining these vertices becoming the line joining these points of tangency; but this line passes through the intersection of the diagonals of the quadrilateral; hence, the diagonals and the lines joining the points of tangency of the opposite sides meet in the same point.

4. *The straight lines joining the points of tangency of a circumscribed triangle with the opposite vertices pass through the same point.*

This becomes evident by moving the alternate vertices of the circumscribed hexagon to the circumference.

526. Exercises.

1. If the straight lines through the corresponding vertices of two triangles meet in the same point, the intersections of their corresponding sides lie in the same straight line (**518**).

2. If the intersections of the corresponding sides of two triangles lie in the same straight line, the straight lines

through their corresponding vertices meet in the same point (**514**).

3. If the three sides of a triangle pass through three fixed points in a straight line, and two vertices of the triangle move in two fixed straight lines, the third vertex moves in a straight line which passes through the intersection of the other two lines (**526**, 2).

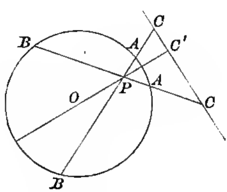
4. If the three vertices of a triangle move in three fixed straight lines which meet in a point, and two sides of the triangle pass through two fixed points, the third side passes through a fixed point which is in the straight line passing through the other two (**526**, 1).

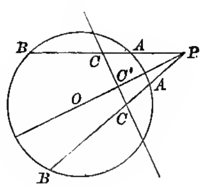
IV. POLE AND POLAR TO THE CIRCLE.

527. Definitions.

1. A point is the *pole* of a fixed line, or a line is the *polar* of a fixed point, with respect to a circle, if the line and point are so situated that the chord of a revolving secant through the point is divided harmonically at the fixed point and the intersection of the fixed line with the secant.

2. The **polar point** is the intersection of the polar with the diameter through the pole.



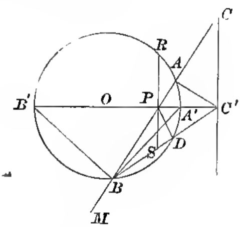


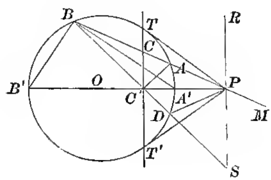
Thus, P is the pole of the fixed line CC, and CC the

polar of the fixed point P, if P and CC are so situated that the chord AB of a revolving secant through P is divided harmonically at P and C (**250,** 3); and C' is the polar point, if PC' passes through the center.

528. Proposition XII.—Theorem.

The polar of a given point with respect to a circle is a straight line perpendicular to the diameter through the point at the harmonic conjugate of the point with respect to the extremities of that diameter.





Let P be the given point; O, the center of the circle; AB, the chord of any secant through P; C, the harmonic conjugate of P with respect to A and B. Draw CC' perpendicular to the diameter $A'B'$ through P, and BC' meeting the circumference in D; also, AC', BA', BB'.

Since $PC'C$ is a right angle, C' is on the circumference whose diameter is PC (**179,** 3); and, since AB is divided harmonically at P and C, $C'P$ bisects the angle $AC'D$ (**260**). Hence, the arcs, AA', $A'D$, are equal; (?) therefore, BA' bisects the angle PBC'; hence, BB', perpendicular to BA', bisects the angle exterior to PBC' (**34,** 4).

Therefore, PC' is divided harmonically at A', B' (**221**), (**223**), (**250,** 3); hence, C' is the harmonic conjugate of P with respect to A', B'; and, since P is fixed, C' is fixed; hence, CC' is the polar of P (**527,** 1).

529. Corollaries.

1. *The radius of the circle is a mean proportional between the distances of the pole and its polar from the center.* (?)

2. *If the pole is without the circle, its polar is the chord joining the points of tangency of the tangents drawn from the pole.* (?)

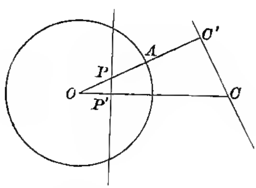
3. *If the pole is on the circumference, its polar is the tangent at the pole.* (?)

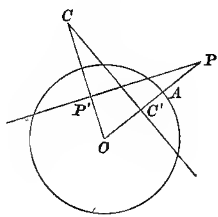
4. *The pole and polar point are interchangeable.* (?)

530. Proposition XIII.—Theorem.

1. *The polar of every point of a straight line passes through the pole of that line.*

2. *The pole of every straight line passing through a given point is on the polar of that point.*





1. Let CC' be a straight line; P, its pole; and C, any point of CC'. OC' through P is perpendicular to the polar CC' (528), and C' is the polar point. Draw PP' perpendicular to OC;

$\therefore \quad OC : OP :: OC' : OP'$; $\therefore$ $OC \times OP' = OP \times OC'$.

But, $OP \times OC' = \overline{OA}^2$ (**529**, 1); $\therefore$ $OC \times OP' = \overline{OA}^2$.

Therefore, PP' through P is the polar of C.

2. Let P be a given point; CC', its polar; and PP', any straight line through P. Draw OC perpendicular to PP'. Then

$$OC \times OP' = \overline{OA}^2.$$

Therefore, C, on the polar of P, is the pole of PP'.

531. Corollaries.

1. *The pole of a straight line is the intersection of the polars of any two of its points.* (?)

2. *The polar of any point is the straight line joining the poles of any two straight lines through that point.* (?)

3. *If the pole is within the circle, its polar is the locus of the intersection of the pair of tangents at the extremities of any chord through the pole.* (?)

4. *If the vertices of one of two polygons are respectively the poles of the sides of the other, the vertices of the second are respectively the poles of the sides of the first.* (?)

5. *If through a fixed point in the plane of a circle any two secants be drawn, and their intersections with the circumference be joined by chords, the locus of the intersections of these chords is the polar of the fixed point.* (?)

V. RECIPROCAL POLARS.

532. Definitions.

1. **Reciprocal polars** are two polygons so related that the vertices of either are respectively the poles of the sides of the other with respect to the same circle, which is called the *auxiliary circle.*

2. A **reciprocal theorem** is a theorem inferred from another by means of reciprocal polars.

533. Proposition XIV.—Problem.

Infer Brianchon's theorem (**524**) *from Pascal's* (**522**).

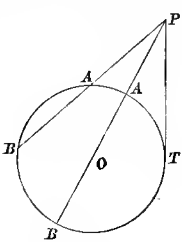
Chords joining the consecutive points of tangency of the sides of the circumscribed hexagon form an inscribed hexagon, the vertices of which are respectively the poles of the sides of the circumscribed hexagon (**529**, 3). Hence, the vertices of the circumscribed hexagon are respectively the poles of the sides of the inscribed hexagon (**531**, 4); therefore, the pole of each of the three diagonals joining opposite vertices of the circumscribed hexagon is on each of two opposite sides of the inscribed hexagon (**530**, 2), and consequently at their intersection. Hence, by Pascal's theorem, the poles of the three diagonals lie on the same straight line; therefore, the three diagonals, which are the polars of these poles, pass through the pole of this straight line (**530**, 1).

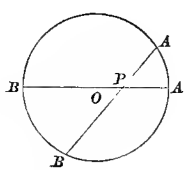
Let Pascal's theorem be inferred from Brianchon's.

VI. RADICAL AXES.

534. Definitions.

1. The **power of a point** in the plane of a circle is the rectangle of the segments, external or internal, into which the point divides the chord passing through it.





Thus, the power of P is $PA \times PB$. The power is

positive or negative, according as the point is without or within the circle, since, in the first case, the segments have like signs, and in the second case, unlike.

2. The **radical axis** of two circles is the locus of the point whose powers with respect to the circles are equal.

535. Proposition XV.—Theorem.

The power of a given point with respect to a given circle is the constant equal to the square of the distance from the point to the center, minus the square of the radius.

In the diagrams (**534**), let p denote the power; d, the distance from the point to the center; and r, the radins.

The power is eqnal to the rectangle of the segments of the chord through the center (**247**), (**245**); then,

$$p = \quad (d - r)\,(d + r) = d^2 - r^2, \text{ 1st diagram;}$$
$$p = -(r - d)\,(r + d) = d^2 - r^2, \text{ 2d diagram.}$$

536. Corollaries.

1. *The power of a point without the circle is equal to the square of the tangent drawn from that point* (**248**).

2. *The power of a point within the circle is equal to minus the square of half the least chord through the point* (**136,** 3), (**246**).

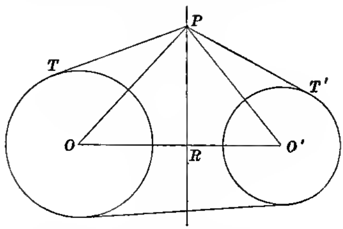
3. *The power of a point on the circumference is equal to zero.* (?)

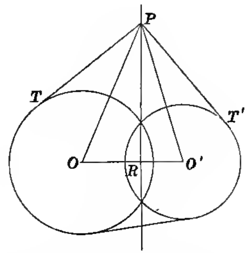
4. *The power of the center is equal to minus the square of the radius.* (?)

537. Proposition XVI.—Theorem.

The radical axis of two circles is the straight line perpendicular to the straight line joining their centers, dividing it so that

the difference of the squares of the segments is equal to the difference of the squares of the radii.





Let O, O', be two circles; r, r', their radii; P, any point of the radical axis; d, d', the distances of P from the centers.

Then, $d^2 - r^2 = d'^2 - r'^2$ (**535**); $\therefore \quad d^2 - d'^2 = r^2 - r'^2$.

Let PR be perpendicular to OO'.

Then, $\overline{OR}^2 = \overline{PO}^2 - \overline{PR}^2$, $\overline{O'R}^2 = \overline{PO'}^2 - \overline{PR}^2$;

$\therefore \quad \overline{OR}^2 - \overline{O'R}^2 = \overline{PO}^2 - \overline{PO'}^2 = d^2 - d'^2 = r^2 - r'^2$.

Hence, PR, perpendicular to OO' at R, is the radical axis, since it embraces any point P of that axis.

538. Corollaries.

1. *There is an infinite number of pairs of circles having the same radical axis and their centers in the same straight line.* (?)

2. *The radical axis of two circles external to each other lies between them, tangent to neither.* (?)

3. *The radical axis of two intersecting circles is the straight line through their points of intersection.* (?)

4. *The radical axis of two circles tangent externally or internally is the common tangent at their point of tangency.* (?)

5. *The radical axis of two circles, one of which lies wholly within the other, is exterior to both circles.* (?)

6. *The tangents to two circles drawn from any point of their radical axis are equal.* (?)

7. *The radical axis bisects any common tangent.* (?)

8. *The radical axes of a system of three circles, taken two and two, meet in a common point called the radical center.* (?)

VII. CENTERS OF SIMILITUDE.

539. Definitions.

1. The **external** and **internal centers of similitude** of two circles are two points which divide the line joining their centers harmonically in the ratio of the two radii.

2. **Homologous points** are the alternate intersections of two circumferences with a transversal through the external center of similitude, or the mean and extreme intersections of two circumferences with the transversal through the internal center of similitude.

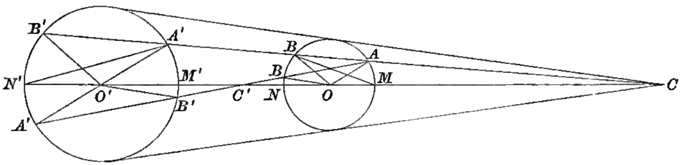
3. **Anti-homologous points** are the extreme and mean intersections of two circumferences with a transversal through the external center of similitude, or the alternate intersections of two circumferences with a transversal through the internal center of similitude.

4. **Anti-homologous chords** are chords joining anti-homologous points of two transversals through the same center of similitude.

540. Proposition XVII.—Theorem.

1. *The transversal through the extremities of parallel radii lying in the same direction passes through the external center of similitude.*

2. *The transversal through the extremities of parallel radii lying in opposite directions passes through the internal center of similitude.*



The similar triangles, COA, $CO'A'$, and $C'OA$, $C'O'A'$, give $CO : CO' :: OA : O'A'$, $C'O : C'O' :: OA : O'A'$; $\therefore$ C, C', are the external and internal centers of similitude.

541. Corollaries.

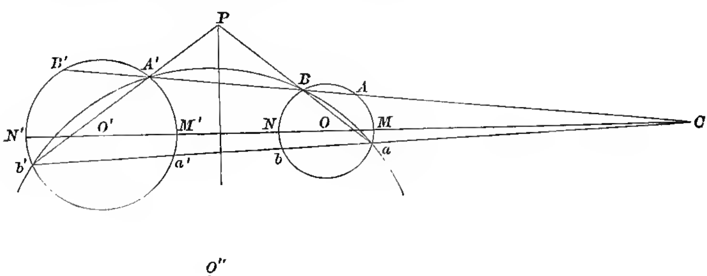
1. *The radii drawn to the intersections of two circumferences with a transversal through either center of similitude are parallel two and two.* (?)

2. *The extremities of parallel radii are homologous points, and tangents at homologous points are parallel.* (?)

3. *The distances from a center of similitude to two homologous points are proportional to the radii of the circles.* (?)

542. Proposition XVIII.—Theorem.

The product of the distances from a center of similitude to two anti-homologous points is constant.



The transversals, CA, CO, intersect the circumferences in the homologous points, A, A', and B, B'; also, M, M', and N, N'.

$$\therefore \quad \frac{CA}{CA'} = \frac{CB}{CB'} = \frac{CM}{CM'} = \frac{OM}{OM'} \quad \textbf{(541, 3)}.$$

But, $CA' \times CB' = CM' \times CN'$ (**247**).

Multiplying the first and second of the above ratios by $CA' \times CB'$, and the third by $CM' \times CN'$, we have

$$CA \times CB' = CA' \times CB = CM \times CN'.$$

But $CM \times CN'$ is constant; $\therefore$ $CA \times CB'$ and $CA' \times CB$ are constant.

543. Corollaries.

1. *The two pairs of anti-homologous points of two transversals through the same center of similitude lie on the same circumference.* (?)

2. *Anti-homologous chords of two circles intersect on their radical axis.* (?)

3. *Tangents at two anti-homologous points in two circles intersect on their radical axis.* (?)

544. Proposition XIX.—Theorem.

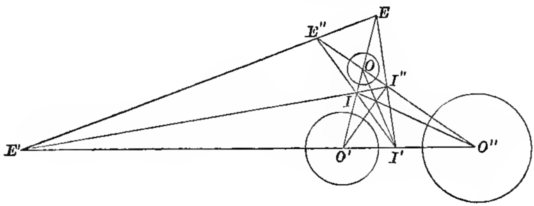
If three circles be taken two and two, then,

1. *The straight lines joining the center of each with the internal center of similitude of the other two meet in a point.*

2. *The three external centers of similitude are in a straight line.*

3. *The external center of similitude of any pair, and the internal centers of similitude of the other pairs, are in a straight line.*

Let O, O', O'', be three circles whose radii are R, R', R''; E, I, and E', I', and E'', I'', respectively, the external and internal centers of similitude of O, O', and O', O'', and O'', O.



1. $\frac{OI}{O'I} = \frac{R}{R'}, \quad \frac{O'I'}{O''I'} = \frac{R'}{R''}, \quad \frac{O''I''}{OI''} = \frac{R''}{R}$ (**539**, 1).

$\therefore \quad \frac{OI \times O'I' \times O''I''}{O'I \times O''I' \times OI''} = \frac{R \times R' \times R''}{R' \times R'' \times R} = 1.$

$\therefore \quad OI \times O'I' \times O''I'' = O'I \times O''I' \times OI''.$

$\therefore \quad OI'$, $O'I''$, $O''I$, meet in a point (**504**, 1).

2. $\frac{OE}{O'E} = \frac{R}{R'}, \quad \frac{O'E'}{O''E'} = \frac{R'}{R''}, \quad \frac{O''E''}{OE''} = \frac{R''}{R}$ (**539**, 1).

$\therefore \quad OE \times O'E' \times O''E'' = O'E \times O''E' \times OE''.$

$\therefore \quad E$, E', E'', are in a straight line (**502**).

This line is called the *external axis of similitude.*

3. $\frac{O''E''}{OE''} = \frac{R''}{R}, \quad \frac{OI}{O'I} = \frac{R}{R'}, \quad \frac{O'I'}{O''I'} = \frac{R'}{R''}$ (**539**, 1).

$\therefore \quad O''E'' \times OI \times O'I' = OE'' \times O'I \times O''I'.$

$\therefore \quad E''$, I, I', are in a straight line (**502**). Likewise, E, I', I'', also, E', I'', I, are in a straight line.

These three lines are called the *internal axes of similitude.*